

LOOP SPACES AND LANGLANDS PARAMETERS

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ABSTRACT. We apply the technique of S^1 -equivariant localization to sheaves on loop spaces in derived algebraic geometry, and obtain a fundamental link between two families of categories at the heart of geometric representation theory. Namely, we categorify the well known relationship between free loop spaces, cyclic homology and de Rham cohomology to recover the category of \mathcal{D} -modules on a smooth stack X as a localization of the category of S^1 -equivariant coherent sheaves on its loop space $\mathcal{L}X$. The main observation is that this procedure connects categories of equivariant \mathcal{D} -modules on flag varieties with categories of equivariant coherent sheaves on the Steinberg variety and its relatives. This provides a direct connection between the geometry of finite and affine Hecke algebras and braid groups, and a uniform geometric construction of all of the categorical parameters for representations of real and complex reductive groups. This paper forms the first step in a project to apply the geometric Langlands program to the complex and real local Langlands programs, which we describe.

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1. INTRODUCTION

In this paper, we apply the technique of S^1 -equivariant localization to sheaves on loop spaces in derived algebraic geometry. Namely, we categorify the well known relation between free loop spaces, cyclic homology and de Rham cohomology into a theorem that recovers the category of \mathcal{D} -modules on a smooth stack X as a localization of the category of S^1 -equivariant coherent sheaves on its loop space $\mathcal{L}X$. This provides a fundamental link between two families of categories at the heart of geometric representation theory.

On the one hand, categories of equivariant \mathcal{D} -modules on flag varieties are central in the representation theory of reductive groups. Consider a complex reductive group G with Lie algebra \mathfrak{g} , Borel subgroup $B \subset G$, and flag variety $\mathcal{B} = G/B$. The localization theory of Beilinson-Bernstein identifies representations of \mathfrak{g} with global sections of (twisted) \mathcal{D} -modules on \mathcal{B} . In particular, highest weight representations are realized by B -equivariant \mathcal{D} -modules on \mathcal{B} , or in other words, by \mathcal{D} -modules on the quotient stack $B \backslash \mathcal{B}$. Given a real form $G_{\mathbf{R}}$ of G with associated symmetric subgroup $K \subset G$, infinitesimal equivalence classes of admissible representations of $G_{\mathbf{R}}$ correspond to Harish-Chandra modules for the pair (\mathfrak{g}, K) . These in turn are realized by K -equivariant \mathcal{D} -modules on \mathcal{B} , or in other words, by \mathcal{D} -modules on the quotient stack $K \backslash \mathcal{B}$. By the work of Adams, Barbasch and Vogan [ABV] and Soergel [So] on the real local Langlands correspondence, (untwisted) \mathcal{D} -modules on $K \backslash \mathcal{B}$ also appear as the Langlands parameters for representations of real forms of the Langlands dual group G^{\vee} . An important unifying structure in the study of all these cases is the natural intertwiner or convolution action of the finite Hecke algebra and braid group carried by these categories.

On the other hand, categories of equivariant coherent sheaves on the Springer resolution, the Steinberg variety, and their relatives have played a prominent role in geometric representation theory since the work of Kazhdan-Lusztig on the p -adic local Langlands conjecture. These categories often arise as geometric versions of representations of affine Hecke algebras and braid groups. For example, a fundamental theorem of Kazhdan-Lusztig identifies the affine Hecke algebra of the Langlands dual group G^{\vee} with the Grothendieck group of equivariant coherent sheaves on the Steinberg variety of G . Recent developments in the tamely ramified geometric Langlands program (in particular, work of Bezrukavnikov and collaborators, see [Be], and Frenkel-Gaitsgory [FG]) have significantly advanced our understanding of the underlying categorical structure. In particular, there are intimate connections with representations of quantum groups, modular representations of Lie algebras, and critical level representations of the loop algebra for the Langlands dual group G^{\vee} .

The main point of this paper is to connect these two families of categories by a categorified, algebro-geometric version of S^1 -equivariant localization. Namely, we explain that the Springer

resolution and Steinberg variety are the derived loop spaces of the quotient stacks $G\backslash\mathcal{B}$ and $B\backslash\mathcal{B}$ respectively. Motivated by the representation theory of real groups we further introduce a collection of spaces, the *Langlands parameter spaces*, which are similarly related to the quotients $K\backslash\mathcal{B}$ by symmetric subgroups. To make all of this precise, we use the formalism of derived algebraic geometry in which it makes sense to discuss the quotient stacks of algebraic geometry and the loop spaces of algebraic topology at the same time. Our main application of this viewpoint is a categorical form of quantization of cotangent bundles which is particularly well adapted to representation theory. Namely, the derived category of quasicoherent sheaves on the loop space of a smooth stack carries a circle action, and the resulting equivariant derived category is expressed in terms of \mathcal{D} -modules on X .¹

In the special case when X is one of the quotient stacks $G\backslash\mathcal{B}$, $B\backslash\mathcal{B}$, or $K\backslash\mathcal{B}$, we recover the derived category of coherent \mathcal{D} -modules on X from coherent sheaves on the Springer resolution, Steinberg variety or Langlands parameter space respectively. This provides a direct connection between finite and affine geometric representation theory of Hecke algebras and braid groups. In the case of the Langlands parameter space, we obtain a uniform geometric construction of all of the categorical parameters for representations of real reductive groups. This is a crucial first step in a project to apply ideas from the geometric Langlands program to the complex and real local Langlands programs.

The paper is organized as follows. In Section 2, we provide a detailed overview of the results of this paper, and in Section 3, an introduction to the intended applications and our general project. In Section 4, we discuss properties of loop spaces and Hochschild (or small loop) spaces in derived algebraic geometry. (The Appendix provides a quick introduction to the theory of derived stacks.) In Section 5, we study equivariant sheaves on derived stacks. Our aim is to explain a categorification of the relation between S^1 -equivariant homology of loop spaces (cyclic homology) and de Rham cohomology. Then in Sections 6 and 7, we recall the role of the flag variety and Steinberg variety in representation theory, introduce the Langlands parameter spaces, and explain the resulting new perspective on the parametrization of Harish-Chandra modules for real groups.

The discussion in Section 3 summarizes a series of papers in preparation [BN1, BN2, BN3] on the representation theory of real reductive groups. Among the results are a conceptual geometric proof of (an affine generalization of) Vogan duality, a canonical equivalence between Harish-Chandra bimodules for Langlands dual complex groups (compatible with Hecke actions), and a derivation of a strengthened form of Soergel's conjecture from a geometric version of the principle of automorphic base change.

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¹It is possible to transport to the setting of derived loop spaces some of the varied structures on free loop spaces in topology. For example, given a stack X , quasicoherent sheaves on its derived loop space $\mathcal{L}X$ form a braided tensor category, a derived form of the *Drinfeld double* of quasicoherent sheaves on X , which is part of a collection of topological field theory operations. We plan to elaborate on this structure in the future.

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2. OVERVIEW

2.1. Derived loop spaces. The free loop space LX of a topological space X comes equipped with many fascinating structures. Of fundamental importance for our purposes is the fact that LX carries a circle action by loop rotation with the constant loops X as fixed points. The theory of equivariant localization for circle actions, relating topology of an S^1 -space and its fixed points, has been applied to spectacular effect in this setting in the work of Witten and many others.

The homology and S^1 -equivariant homology of LX are intimately related to the Hochschild homology and cyclic homology of cochains on X respectively. In general, Hochschild homology and cyclic homology provide a purely algebraic or categorical approach to the geometry of free loop spaces. When applied to commutative rings or schemes, Hochschild homology captures the algebra of differential forms, while cyclic homology captures de Rham cohomology. In other words, cyclic homology allows us to view the de Rham complex of a scheme as an algebraic analogue of S^1 -equivariant cochains on a free loop space.

The above interpretation of de Rham cohomology is quite familiar in mathematical physics, or more specifically, in supergeometry. Namely, consider the cohomology of the circle $H^*(S^1, \mathbf{C}) = \mathbf{C}[\eta]$ (with η of degree one) as the supercommutative ring of functions on the odd line $\mathbf{C}^{0|1}$. For a smooth scheme X (likewise for smooth manifolds), the mapping space $\mathrm{Map}(\mathbf{C}^{0|1}, X)$ is the superscheme given by the odd tangent bundle $\mathbb{T}_X[-1]$ (we will remember its \mathbf{Z} -grading of one). Thus one may think of $\mathbb{T}_X[-1]$ as a linearized analogue of the free loop space. Observe that functions on $\mathbb{T}_X[-1]$ are simply differential forms $\Omega_X^\bullet = \mathrm{Sym}^\bullet \Omega_X[1]$ placed in negative (homological) degrees. The analogue of the S^1 -action on the loop space is translation along $\mathbf{C}^{0|1}$. This action defines a canonical, square zero, odd vector field on $\mathbb{T}_X[-1]$ which is easily seen to be the de Rham differential considered as an odd derivation of Ω_X^\bullet .

Another point of view on the same construction is to model the circle by two points connected by two line segments, and to interpret concretely what maps from such an object to X should be. First, mapping two points to X defines the product $X \times X$. One of the line segments connecting the points says the points are equal: we should impose the equation $x = y$ that defines the diagonal $\Delta \subset X \times X$. Then the other line segment says the points are equal again, so we should impose the equation $x = y$ again, or in other words, take the self-intersection $\Delta \cap \Delta \subset X \times X$. Of course, we could interpret this naively as being a copy of X again, however the intersection is far from transverse. For \mathcal{O}_X the ring of functions on X , the tensor product $\mathcal{O}_X \otimes_{\mathcal{O}_{X \times X}} \mathcal{O}_X$ needs to be derived (there are higher Tor terms). If we use the Koszul complex to calculate the derived tensor product $\mathcal{O}_X \otimes_{\mathcal{O}_{X \times X}}^\mathbb{L} \mathcal{O}_X$, we again find the commutative differential graded ring Ω_X^\bullet . Thus if we accept the broader framework of supergeometry as a correction for the degenerate intersection, then we again discover $\mathbb{T}_X[-1]$ as a model for the loop space of X .

We would like to apply a version of the above picture in more complicated contexts where the geometry of a smooth variety X is replaced by the equivariant geometry of X with respect to an algebraic group action. Our motivating examples are flag varieties equipped with the action of various groups of significance in representation theory. Thus we would like to work

in a context that generalizes schemes simultaneously in two directions. We need to be able to perform the following operations without fear of missing some aspect of the geometry:

- Quotients (and more general gluings or colimits) of schemes
- Intersections (and more general fiber products or limits) of schemes

In both cases, passing to derived versions of the above operations guarantees that no information is lost. The need to correct quotients has long been recognized in algebraic geometry, and a powerful and flexible solution is provided by the theory of stacks. Within the framework of this theory, there is an interesting formal substitute for the loop space of a stack X . Namely, we can consider the classifying stack $pt/\mathbf{Z} = B\mathbf{Z}$ as a version of the circle, and define the *inertia space* of X as the mapping stack

$$IX = \mathrm{Hom}(B\mathbf{Z}, X).$$

Since the circle $B\mathbf{Z}$ is purely homotopical, it cannot map nontrivially along the scheme direction of X but only in the stacky direction. As a result, the objects of IX are objects of X equipped with automorphisms. In particular for the stack $BG = pt/G$, we find that $I(BG)$ is the adjoint quotient G/G . Note that the inertia stack IX of a scheme X is nothing more than X again, so we have not yet accounted for the excess self-intersection of the diagonal.

The need to correct for degenerate intersections has come to prominence more recently, in particular through the derived moduli spaces vision of Drinfeld, Deligne, Feigin and Kontsevich [K2], and in particular the theory of virtual fundamental classes (see [BeF, Be, CFK] and references therein). Derived intersections and fiber products are also necessary tools in geometric representation theory since categories of sheaves are sensitive to derived structures. In recent years, through the work of Toën-Vezzosi, Lurie and others, the foundations of a theory of *derived schemes and stacks* have emerged, elegantly combining algebraic geometry and homotopical algebra. In broad outline, one replaces commutative rings by simplicial commutative rings (or connective commutative differential graded rings, in characteristic zero), and functors of points take values not in sets but in topological spaces. (One of the formal but complicated aspects of the story is that both the domain and target of such functors must be treated with the correct homotopical understanding.) For the reader's convenience, we provide an overview of the theory of derived stacks in the Appendix, and we recommend Toën's survey [To2] for more details and references.

The framework of derived stacks allows one to correctly take quotients (since functors of points take values in topological spaces), and to correctly form intersections (since functors of points are defined on simplicial commutative rings). One can immediately import all of the structures of homotopy theory to this setting. In particular, we can look at the derived stack of maps from $S^1 = B\mathbf{Z}$ to another stack X . This gives an enhanced version of the inertia stack which we simply call the *loop space*

$$\mathcal{L}X = \mathbf{R}\mathrm{Hom}(S^1, X).$$

It combines the notion of odd tangent bundle and inertia stack: $\mathcal{L}X = \mathbb{T}_X[-1]$, for X a smooth scheme, and $\mathcal{L}(BG) = G/G$ for the classifying space of an algebraic group G . In general, for X a smooth scheme equipped with a G -action, the loop space $\mathcal{L}(X/G)$ combines even directions coming from stabilizers of orbits in X , and odd directions coming from normals to the orbits.

Inside of the loop space $\mathcal{L}X$, we single out the small loop space or Hochschild space $\mathcal{H}X$. By definition, it is the formal completion of $\mathcal{L}X$ along the constant loops $X \subset \mathcal{L}X$. For a smooth stack, we show that $\mathcal{H}X$ is the formal completion of the shifted tangent bundle $\mathbb{T}_X[-1]$ along its zero section. In particular, $\mathcal{H}(BG) = \widehat{G}/G$ is the adjoint quotient of the formal group, while

$\mathcal{H}X = \mathcal{L}X$ for X a smooth scheme. The key motivation for introducing $\mathcal{H}X$ is that small loops are local objects on X in a suitable sense.

Although not needed in this paper, it is interesting to revisit (Section 4.4) some of the familiar structures on loop spaces and Hochschild chains in the setting of a smooth scheme X . Namely, $\mathcal{L}X$ forms a family of groups over X (in an appropriate homotopical sense), and $\mathcal{H}X$ is the corresponding family of formal groups. Functions on $\mathcal{H}X$ are simply the Hochschild chains of X , and the Hochschild-Kostant-Rosenberg theorem then plays the role of the Baker-Campbell-Hausdorff theorem (as explained in [M] and expanded on in [R1, R2, RW]). It identifies the formal completion of the Lie algebra given by the odd tangent bundle $\mathbb{T}_X[-1]$ with the formal group $\mathcal{H}X$ itself. The Lie algebra structure on $\mathbb{T}_X[-1]$ is given by the Atiyah class as explained in [Ka2]. The Hochschild cochains are the enveloping algebra of $\mathbb{T}_X[-1]$, and hence may be thought of as distributions supported on constant loops inside small loops. Thus Hochschild cochains are analogues of the algebras of chiral differential operators of Malikov, Schechtman and Vaintrob, while the composition of loops is analogous to the factorization structure on the small formal loop space of Kapranov and Vasserot.

2.2. Sheaves on loop spaces. We are primarily interested in the loop space $\mathcal{L}X$ of a stack for its derived category of quasicoherent sheaves (or rather its differential graded (dg) enhancement) which we denote by $L_{\text{qcoh}}(\mathcal{L}X)$.² In order to avoid technical complications we will work throughout with X a smooth Artin (1-)stack with affine diagonal, though much of the paper could be generalized to higher stacks and presumably with proper care to singular and derived targets as well. We will focus on the dg derived category of quasicoherent sheaves with coherent cohomology which we denote by $L_{\text{coh}}(\mathcal{L}X)$.

The loop space $\mathcal{L}X$ automatically carries an action of the group S^1 , which may be expressed for example through the Connes formalism of cyclic objects. We may consider S^1 -equivariant sheaves on the loop space $\mathcal{L}X$, or equivalently, sheaves on the space of unparametrized loops $\mathcal{L}X/S^1$. The S^1 -action is also manifested as an automorphism of the identity functor of $L_{\text{qcoh}}(\mathcal{L}X)$, which on a sheaf \mathcal{F} is given by the monodromy of \mathcal{F} along the S^1 -orbits. Roughly speaking, equivariant sheaves are given by sheaves on $\mathcal{L}X$ equipped with a homotopy between their monodromy operator and the identity. On the other hand, on the space of small loops $\mathcal{H}X \simeq \widehat{\mathbb{T}}[-1]$ we have the odd derivation given by the de Rham differential. To relate the two, we discuss a general Koszul duality formalism for equivariant sheaves in Section 5.2. In the case of S^1 acting on a point, this gives rise to the well-known equivalence between cyclic vector spaces and complexes with exterior algebra action (mixed complexes). After restricting to the Hochschild space $\mathcal{H}X$ of small loops, we check that an S^1 -equivariant structure is precisely a lifting of the odd vector field on $\mathbb{T}_X[-1]$ to the sheaf. In other words, the sheaf becomes endowed with an action of the de Rham differential – considered as a *homotopy* on the *homological* complex $\Omega_X^{-\bullet}$ – or equivalently an action of the algebra $\Omega_X^{-\bullet}[d]$ in which we’ve adjoined d in degree -1 . We thus have the following categorification of the relation between de Rham cohomology and cyclic homology:

²By general formalism, the category $L_{\text{qcoh}}(\mathcal{L}X)$ inherits the rich structures of loop spaces. The theory of string topology provides an appealing way to organize many of these structures. It describes the natural operations on the homology of loop spaces as a part of two-dimensional topological field theory. In the setting of derived algebraic geometry, one can show that $L_{\text{qcoh}}(\mathcal{L}X)$ possesses a categorified version of the string topology operations carried by the homology of loop spaces. In particular, the pair of pants defines a braided tensor structure on $L_{\text{qcoh}}(\mathcal{L}X)$ (more precisely, an E_2 -category structure). This generalizes the notion of the usual Drinfeld double of G whose modules are $L_{\text{qcoh}}(\mathcal{L}(BG)) = L_{\text{qcoh}}(G/G)$.

Theorem 2.1 (Theorem 5.4 below). *For a smooth Artin stack X , there is a canonical quasi-equivalence of dg derived categories*

$$L_{\text{qcoh}}(\mathcal{H}X)^{S^1} \simeq L_{\text{qcoh}}(X, \Omega_X^{-\bullet}[d])$$

preserving subcategories of coherent objects.

Modules over the de Rham complex are intimately related to sheaves with flat connection, or \mathcal{D} -modules, on X . As explained in [Ka1, BD] the relation between differential operators \mathcal{D}_X and the de Rham complex (Ω_X, d) is a form of Koszul duality.³ First, a form of Koszul duality identifies $\Omega_X^{-\bullet}[d]$ -modules with dg modules over the Rees algebra \mathcal{R}_X of \mathcal{D}_X placed in positive even degrees.

To recover the category of \mathcal{D}_X -modules, or equivalently dg modules over the de Rham complex (Ω_X^{\bullet}, d) , we pass to the *periodic* version, by tensoring the category of modules with the ring $\mathbf{C}[u, u^{-1}]$. On the other side, this amounts to performing the usual localization of S^1 -equivariant cohomology. Observe that S^1 -equivariant sheaves form a category linear over the ring $H_{S^1}^*(pt) = \mathbf{C}[u]$, with $|u| = 2$. If we invert u , we obtain a $\mathbf{Z}/2\mathbf{Z}$ -periodic dg category

$$L_{\text{qcoh}}(\mathcal{H}X)_{\text{per}}^{S^1} = L_{\text{qcoh}}(\mathcal{H}X)^{S^1} \otimes_{\mathbf{C}[u]} \mathbf{C}[u, u^{-1}].$$

The inversion of u matches up with the localization from the Rees algebra to \mathcal{D}_X itself, resulting in the following relation between \mathcal{D}_X -modules and the periodic cyclic category of X :

Corollary 2.2. *For a smooth Artin stack X , there are canonical quasi-equivalences of dg derived categories of coherent sheaves*

$$\begin{aligned} L_{\text{coh}}(\mathcal{H}X)^{S^1} &\simeq L_{\text{coh}}(X, \mathcal{R}_X) \\ L_{\text{coh}}(\mathcal{H}X)_{\text{per}}^{S^1} &\simeq L_{\text{coh}}(X, \mathcal{D}_X)_{\text{per}} \end{aligned}$$

This picture of \mathcal{D} -modules as S^1 -equivariant sheaves on the Hochschild space is of course closely related to many other ways to express flat connections. Most directly, the dg Lie algebra $\mathbb{T}_X[-1]$ acts by endomorphisms of the identity of $L_{\text{qcoh}}(X)$. Namely, the action $\mathbb{T}_X[-1] \otimes \mathcal{F} \rightarrow \mathcal{F}$ is given by the Atiyah class of \mathcal{F} , which is the one-jet extension $\mathcal{J}\mathcal{F} \in \text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \Omega_X)$. A trivialization of the Atiyah class of a sheaf is precisely the data of a connection. The structure of trivialization of the monodromy on sheaves on $\mathcal{H}X$ is related by pullback to trivialization of the Atiyah class on X . This gives an alternative route to recover the relation between S^1 -equivariant sheaves on $\mathcal{H}X$ and flat connections on X .

For another point of view, note that Koszul duality identifies sheaves on the odd tangent bundle (modules for $\Omega_X^{-\bullet}$) with sheaves on the cotangent bundle (modules for $\text{Sym}^{\bullet} \mathbb{T}_X$). Passing from the graded ring $\Omega_X^{-\bullet}$ to the differential graded ring $(\Omega_X^{-\bullet}, d)$ corresponds to deforming the graded ring $\text{Sym}^{\bullet} \mathbb{T}_X$ to the filtered ring \mathcal{D}_X . Our original motivation for this story came from the observation that in applications to representation theory, it is often easier to identify the differential d than the deformation quantization \mathcal{D}_X . Thus we think of loop spaces and their circle action as a useful geometric counterpart to cotangent bundles and their quantization. The same paradigm appears in relating string topology of X (that is, topology of LX) to the A-model (Fukaya category) of T^*X . In that sense, this picture is a counterpart to the emerging relation between Fukaya categories and \mathcal{D} -modules or constructible sheaves (see [KW, NZ, N]).

³Koszul duality does not naively give an equivalence on categories of quasicohherent sheaves. This can be corrected by modifying the notion of equivalence of de Rham modules (as in [BD]), or by killing some large \mathcal{D} -modules which are missed by the de Rham functor. In either case, the categories of coherent modules, which are our primary interest, are unaffected.

We would also like to mention the beautiful work of Simpson and Teleman [ST] on de Rham's theorem on stacks. It deals with \mathcal{D} -modules on general stacks as sheaves equivariant for the formal neighborhood of the diagonal. This natural picture is related to our loop space picture by a somewhat awkward shift of grading (taking the homological or simplicial ring given by Hochschild homology and translating it into a cohomological or cosimplicial ring giving the usual de Rham stack). On the categorical level, this requires lifting to a “mixed” setting à la [BGS]. We hope to return to this issue in the future.

2.3. Equivariant \mathcal{D} -modules on flag varieties. We now turn to representation theory, which is our primary impetus for this work and which takes up the last two sections of the paper. Our motivation is the observation that (the equivariant versions of) the Steinberg variety and its relatives, the Langlands parameter spaces, are loop spaces, and that the corresponding equivariant localization (as described above) matches well with the Langlands program. In this section and the next we review some of the background in geometric representation theory, with a large emphasis on representations of real Lie groups. Our results are described in Section 2.5. We recommend that the reader interested in applications in geometry or complex groups skip the real material on first reading (see Section 2.4.1 for the definition of the Langlands parameter spaces).

Let G be a complex reductive group with Lie algebra \mathfrak{g} , Borel subgroup $B \subset G$, and flag variety $\mathcal{B} = G/B$. A primary motivation for studying \mathcal{D} -modules on algebraic stacks comes from the localization theory of Beilinson-Bernstein. It identifies representations of \mathfrak{g} with global sections of twisted \mathcal{D} -modules on \mathcal{B} . Furthermore, given a subgroup $K \subset G$, it identifies modules for the Harish-Chandra pair (\mathfrak{g}, K) with global sections of K -equivariant twisted \mathcal{D} -modules on \mathcal{B} . The following well-known cases are of traditional interest:

(1) The case $K = G$ gives the Borel-Weil description of irreducible algebraic (equivalently, finite-dimensional) representations of G as sections of equivariant line bundles on \mathcal{B} .

(2) The case $K = B$ gives highest-weight modules as sections of twisted \mathcal{D} -modules smooth along the Schubert stratification. Such modules are closely related to the objects of category \mathcal{O} of Bernstein-Gelfand-Gelfand. Via the identification

$$B \backslash \mathcal{B} = G \backslash (\mathcal{B} \times \mathcal{B}),$$

they are also closely related to Harish-Chandra bimodules, and thus also to admissible representations of G considered as a real Lie group.

(3) Finally, the case of a symmetric subgroup $K \subset G$ is of fundamental interest for its relation to real groups. By a symmetric subgroup, we mean the fixed points of an algebraic involution η . Such involutions correspond to antiholomorphic involutions θ and hence real forms $G_{\mathbf{R}}$ of G via the assignment $\theta = \eta \circ \kappa$ where κ is a commuting Cartan involution of G . In this case, the irreducible K -equivariant twisted \mathcal{D} -modules give infinitesimal equivalence classes of irreducible admissible representations of $G_{\mathbf{R}}$.

In this paper, we will restrict our attention to untwisted \mathcal{D} -modules. This choice reflects the fact that we intend to apply the ideas of this paper to \mathcal{D} -modules in their role as Langlands parameters. According to Adams-Barbasch-Vogan [ABV] and Soergel [So], it is untwisted \mathcal{D} -modules which arise in this way. In addition, rather than abelian categories, we will work with the corresponding K -equivariant derived categories $\mathcal{D}_K(\mathcal{B})$ of \mathcal{D} -modules on \mathcal{B} for each of the above subgroups K . (Our convention implicit in the notation $\mathcal{D}_K(\mathcal{B})$ will be to consider only coherent \mathcal{D} -modules.) It is important to note that the equivariant categories are *not* the

derived categories of the abelian categories. For example, there are higher Ext groups between equivariant sheaves as can be seen already in the case $K = G = \mathbb{G}_m$ where we have

$$\mathcal{D}_K(\mathcal{B}) = \mathcal{D}_{\mathbb{G}_m}(pt) = H_{S^1}^*(pt) - \text{mod.}$$

In the case $K = B$, the category $\mathcal{D}_B(\mathcal{B})$ is a monoidal category under convolution, which acts on any category of the form $\mathcal{D}_K(\mathcal{B})$. The action gives rise to an action on $\mathcal{D}_K(\mathcal{B})$ of the Artin braid group of G , generalizing the classical principal series intertwining operators. The Koszul duality theorem of Beilinson-Ginzburg-Soergel [BGS] identifies $\mathcal{D}_B(\mathcal{B})$ with the derived category of Harish-Chandra bimodules with unipotent generalized infinitesimal character.

The K -equivariant derived categories $\mathcal{D}_K(\mathcal{B})$ for a symmetric subgroup $K \subset G$ play an important role in the local Langlands program over the real numbers [ABV, So]. In [ABV], Adams, Barbasch and Vogan recast the parametrization of irreducible admissible representations of real forms of a complex reductive group G^\vee (in the form it developed from the works of Harish-Chandra, Langlands, Shelstad and others) in terms of these categories.⁴ They reinterpret Vogan's character duality [V] as an extension of this classification to the Grothendieck groups of such categories.

To explain the general shape of this picture, we introduce some further notation. Fix once and for all an algebraic involution η of the complex reductive group G . Then associated to η is a finite collection $\Theta(\eta)$ of antiholomorphic involutions of the Langlands dual group G^\vee . For each $\theta \in \Theta(\eta)$, we write $G_{\mathbf{R},\theta}^\vee \subset G^\vee$ for the corresponding real form.

Let \mathfrak{h} denote the universal Cartan algebra of \mathfrak{g} , and let W denote its Weyl group. For each $[\lambda] \in \mathfrak{h}/W \simeq (\mathfrak{h}^\vee)^*/W$, we write $\mathcal{HC}_{\theta,[\lambda]}$ for the category of Harish-Chandra modules for the real form $G_{\mathbf{R},\theta}^\vee$ with generalized infinitesimal character given by $[\lambda]$.

For simplicity, we will restrict our attention to the case when $[\lambda]$ is regular. Fix a semisimple lift $\lambda \in \mathfrak{g}$ of the infinitesimal character $[\lambda]$, and let $\alpha \in G$ denote the element $\exp(\lambda)$. Let $G_\alpha \subset G$ be the reductive subgroup that centralizes α , and let $\mathcal{B}_\alpha = G_\alpha/B_\alpha$ be its flag variety. Consider the finite set of twisted conjugacy classes

$$\Sigma(\eta, \alpha) = \{\sigma \in G \mid \sigma\eta(\sigma) = \alpha\}/G.$$

Each $\sigma \in \Sigma(\eta, \alpha)$ defines an involution of G_α , and we write $K_{\alpha,\sigma} \subset G_\alpha$ for the corresponding symmetric subgroup.

Theorem 2.3 ([V, ABV]). *There is a perfect pairing between the Grothendieck groups*

$$\bigoplus_{\theta \in \Theta(\eta)} K(\mathcal{HC}_{\theta,[\lambda]}) \quad \leftrightarrow \quad \bigoplus_{\sigma \in \Sigma(\eta, \alpha)} K(\mathcal{D}_{K_{\alpha,\sigma}}(\mathcal{B}_\alpha))$$

respecting Hecke symmetries.

In [ABV], this theorem is combined with the microlocal geometry of \mathcal{D} -modules (in particular, the geometry of the cotangent bundles $T^*(K_{\alpha,\sigma} \backslash \mathcal{B}_\alpha)$) to study Arthur's conjectures.

Soergel [So] extended this K -theoretic picture to the categorical level, conjecturing the existence of a (Koszul duality) equivalence of derived categories of Harish-Chandra modules and derived categories of \mathcal{D} -modules on the corresponding geometric parameter spaces $K_{\alpha,\sigma} \backslash \mathcal{B}_\alpha$. A form of Soergel's conjecture reads as follows. To state it, we write $\mathcal{HC}_{\theta,[\lambda]}^{pro}$ for the abelian category obtained by pro-completing $\mathcal{HC}_{\theta,[\lambda]}$ with respect to the generalized infinitesimal character.

⁴Note the nonstandard switching of the roles of G and its Langlands dual group G^\vee . This notational choice is (partially) excused by the fact that this paper takes place completely on the spectral side of Langlands duality.

Conjecture 2.4 ([So]). There is an equivalence of derived categories

$$\bigoplus_{\theta \in \Theta(\eta)} \mathcal{D}(\mathcal{HC}_{\theta, [\lambda]}^{pro}) \simeq \bigoplus_{\sigma \in \Sigma(\eta, \alpha)} \mathcal{D}_{K_{\alpha, \sigma}}(\mathcal{B}_{\alpha})$$

Soergel establishes this conjecture in the case of tori, SL_2 and most importantly for complex groups G^{\vee} (considered as real forms of their complexifications).

As mentioned in [ABV], it is important to find a way to fit together the geometric parameter spaces $K_{\alpha, \sigma} \backslash \mathcal{B}_{\alpha}$ for varying α . In particular, such a family is necessary if one hopes to have a uniform picture for representations with different infinitesimal characters. One of the outcomes of this work is a solution to this problem. As we will see, the geometric parameter spaces naturally emerge from the loop geometry of Steinberg varieties.

2.4. Equivariant coherent sheaves on Steinberg varieties. Consider a complex reductive group G with Borel subgroup $B \subset G$, maximal unipotent subgroup $U \subset B$, universal Cartan $H = B/U$, and flag variety $\mathcal{B} = G/B$.

We will use the name Grothendieck-Springer simultaneous resolution for the smooth scheme \tilde{G} that parametrizes pairs (g, B) of an element $g \in G$ and a Borel subgroup $B \in \mathcal{B}$ such that $g \in B$. We will always think of \tilde{G} as a family over the universal Cartan H via the canonical projection

$$(g, B) \mapsto [g] \in B/U.$$

The fiber over the identity $e \in H$ is the usual Springer simultaneous resolution \tilde{G}_e that parametrizes pairs (u, B) of a Borel subgroup $B \in \mathcal{B}$ and a unipotent element $u \in B$. Its two canonical projections exhibit \tilde{G}_e on the one hand as the cotangent bundle $T^*\mathcal{B}$, and on the other hand as a smooth resolution of the unipotent cone of G .

We will use the name Steinberg variety for the scheme \mathcal{St} that parametrizes triples (g, B_1, B_2) of a pair of Borel subgroups $B_1, B_2 \in \mathcal{B}$ and an element in their intersection $g \in B_1 \cap B_2$. We will always think of \mathcal{St} as a family over the product of two copies of the universal Cartan $H \times H$ via the canonical projection

$$(g, B_1, B_2) \mapsto ([g]_1, [g]_2) \in B_1/U_1 \times B_2/U_2$$

Its image consists of pairs of elements which are related by the Weyl group action. The fiber over the identity $(e, e) \in H \times H$ is the usual Steinberg variety $\mathcal{St}_{e, e}$ that parametrizes triples (u, B_1, B_2) of a pair of Borel subgroups $B_1, B_2 \in \mathcal{B}$ and a unipotent element in their intersection $u \in B_1 \cap B_2$. In general, the Steinberg variety \mathcal{St} is connected, but has irreducible components labeled by the Weyl group, and hence as long as G is nonabelian, \mathcal{St} is singular.

The fundamental relationship between the Grothendieck-Springer simultaneous resolution \tilde{G} and the Steinberg variety \mathcal{St} is that the latter is given by the fiber product

$$\mathcal{St} = \tilde{G} \times_G \tilde{G}.$$

Because the projection $\tilde{G} \rightarrow G$ is semi-small, the derived fiber product coincides with the above naive fiber product. By the usual formalism of correspondences, this implies that coherent sheaves on \mathcal{St} form a convolution algebra which acts on coherent sheaves on \tilde{G} (see [CG] for a detailed exposition). The importance of this construction in representation theory derives from the work of Kazhdan-Lusztig on the tamely ramified p -adic local Langlands program (the Deligne-Langlands conjecture) [KL]. The starting point of this theory is their identification of the Grothendieck group of $(G \times \mathbf{C}^{\times})$ -equivariant coherent sheaves on the fiber $\mathcal{St}_{e, e}$ with the affine Hecke algebra of the Langlands dual group G^{\vee} . As a result, all modules over the

affine Hecke algebra admit a spectral description in terms of various Grothendieck groups of equivariant coherent sheaves on the Springer resolution.

More recently, the equivariant derived categories of coherent sheaves underlying the above K -theoretic story have begun to be understood by the work of Bezrukavnikov and collaborators (including Arkhipov, Ginzburg, Mirkovic, and Rumynin; see [Be] for an overview), the work of Frenkel-Gaiitsgory ([FG], see [F] for an overview) and the work of Gukov-Witten [GW]. These advances have a wide range of applications to modular representation theory and the Lusztig conjectures, representation theory of quantum groups, representations of affine algebras at the critical level, and the local geometric Langlands conjecture. In a striking categorification of the work of Kazhdan-Lusztig, Bezrukavnikov has identified the equivariant derived category of the Steinberg variety (as a monoidal differential graded category) with the affine Hecke category of \mathcal{D} -modules on the affine flag variety of the Langlands dual group G^\vee . One immediate consequence is that the equivariant derived categories of Springer fibers carry actions of the affine Hecke category, and hence of the affine braid group. More generally, all module categories over the affine Hecke category admit a spectral description in terms of equivariant coherent sheaves on Springer fibers.

2.4.1. Langlands parameter spaces. In an ongoing project to better understand representations of real groups, Vogan duality, and Soergel's conjecture, the fundamental objects that arise are certain (automorphic) module categories for the affine Hecke category. By definition, a successful characterization of these categories would involve their spectral description as module categories over equivariant coherent sheaves on the Steinberg variety. Thanks to various structures on these categories, it is possible to guess what form this spectral description should take.

As in Vogan duality and Soergel's conjecture, our starting point is a fixed algebraic involution η of the group G . In Section 7, given such an involution η , we introduce a scheme St^η which we call the Langlands parameters space. By construction, it parametrizes pairs consisting of a Borel subgroup $B \subset G$ and an element $g \in G$ whose η -twisted square is contained in B :

$$St^\eta = \{(g, B) \in G \times \mathcal{B} \mid g\eta(g) \in B\}.$$

The group G naturally acts on St^η by twisted conjugation. By the general formalism of correspondences, equivariant coherent sheaves on St^η form a module category over equivariant coherent sheaves on the Steinberg variety St .

One of the primary aims of this paper is to explain the close relationship between St^η and the geometric parameter spaces appearing in Vogan duality and Soergel's conjecture. In particular, we will see that \mathcal{D} -modules on the geometric parameter spaces can be recovered from equivariant coherent sheaves on St^η . Furthermore, the form of this relationship can be transported back to the original (automorphic) module categories for which equivariant coherent sheaves on St^η should provide a spectral description. Namely, there is a precise form in which the loop spaces of the geometric parameter spaces and their S^1 -equivariant geometry can be interpreted in terms of equivariant coherent sheaves on St^η and their intrinsic categorical structures.

2.5. Langlands parameters as loop spaces. The central theme of this paper is that the fundamental relationship between the equivariant geometry of the flag variety \mathcal{B} and that of the Springer variety \tilde{G} and Steinberg variety St is given by the formalism of loop spaces.

To begin, observe that the quotient of the flag variety \mathcal{B} by the group G is nothing more than the classifying stack pt/B of a Borel subgroup $B \subset G$. Thus the loop space $\mathcal{L}(pt/B)$ is immediately seen to be the adjoint quotient B/B . But this is precisely the equivariant Springer variety

$$\tilde{G}/G \simeq \mathcal{L}(G \backslash \mathcal{B}).$$

We observe that this simple statement generalizes to the Steinberg variety.

Theorem 2.5. *There is a canonical isomorphism of derived stacks*

$$\mathcal{S}t/G \simeq \mathcal{L}(B \backslash \mathcal{B}) \simeq \mathcal{L}(G \backslash (\mathcal{B} \times \mathcal{B})).$$

Via this statement, we can transport all of the structures of loop spaces to the equivariant Steinberg variety. For example, it follows that $\mathcal{S}t/G$ carries a circle action, and the derived category of quasicoherent sheaves on $\mathcal{S}t/G$ carries a braided monoidal structure and other string topology operations.

Our primary application of the above theorem follows from restricting to small loops and applying S^1 -equivariant localization. In this way, we recover the category of B -equivariant \mathcal{D} -modules on the flag variety. More generally, by replacing small loops by alternative formal completions, we obtain categories of Borel equivariant \mathcal{D} -modules on flag varieties for various subgroups.

For any $(\alpha, \beta) \in H \times H$, we write $\mathcal{S}t_{\alpha, \beta}$ for the inverse image in $\mathcal{S}t$ of the formal neighborhood of (α, β) under the canonical projection. Note that $\mathcal{S}t_{\alpha, \beta}$ is nonempty if and only if α and β are related by the Weyl group. In particular, we have the formal Steinberg variety $\mathcal{S}t_{e, e}$ corresponding to the identity $(e, e) \in H \times H$.

Finally, for any $\alpha \in H$, we write $G_\alpha \subset G$ for the reductive subgroup that centralizes α , and $\mathcal{B}_\alpha = G_\alpha/B_\alpha$ for its flag variety.

Theorem 2.6. *There is a canonical quasi-equivalence of periodic dg derived categories*

$$L_{\text{coh}}(\mathcal{S}t_{e, e}/G)_{\text{per}}^{S^1} \simeq \mathcal{D}_B(\mathcal{B}) \otimes_{\mathbf{C}} \mathbf{C}[u, u^{-1}].$$

More generally, for any $\alpha \in H$, and Weyl group element w , there are canonical quasi-equivalences of periodic dg derived categories

$$L_{\text{coh}}(\mathcal{S}t_{\alpha, w\alpha}/G)_{\text{per}}^{S^1} \simeq \mathcal{D}_{B_\alpha}(\mathcal{B}_\alpha) \otimes_{\mathbf{C}} \mathbf{C}[u, u^{-1}].$$

It is worth pointing out that the theorem is not a direct consequence of our previous results on the relation between S^1 -equivariant sheaves on Hochschild spaces and \mathcal{D} -modules. For example, the adjoint quotient $\mathcal{S}t_{e, e}/G$ is not the Hochschild space of $B \backslash \mathcal{B}$, but rather also contains loops which are large in the unipotent direction. What we have is an S^1 -equivariant embedding

$$\mathcal{H}(B \backslash \mathcal{B}) \hookrightarrow \mathcal{S}t_{e, e}/G.$$

One can show that restricting coherent sheaves along this embedding gives an equivalence and then the theorem follows from our previous results. A similar argument holds for a general parameter α .

An interesting aspect of the theorem is the general “discontinuity” of the objects appearing on the right hand side. From a geometric perspective, the quotients $B_\alpha \backslash \mathcal{B}_\alpha$ do not form a nice family as we vary the parameter α . But the theorem says that the loop spaces of these quotients do fit into the nice family formed by the equivariant Steinberg variety $\mathcal{S}t/G$. Here we should emphasize that we are thinking about $\mathcal{S}t/G$ as a loop space, rather than thinking about it along the more traditional lines of a cotangent bundle. In the discussion to follow, we describe similar results for \mathcal{D} -modules on the geometric parameter spaces for real reductive groups. In that context, it is only the loop spaces which fit together into a nice family, not the cotangent bundles.

Now fix an algebraic involution η of the group G . Our aim is to describe a generalization of the above results for the Langlands parameter space $\mathcal{S}t^\eta$ and the geometric parameter spaces $K_{\alpha, \sigma} \backslash \mathcal{B}_\alpha$ appearing in Vogan duality and Soergel’s conjecture. The basic observation that

underlies all applications is that the equivariant Langlands parameter space \mathcal{St}^η/G can be naturally identified as a path space.

Theorem 2.7. *The equivariant Langlands parameter space \mathcal{St}^η/G is the derived space of paths*

$$\gamma = (\gamma_1, \gamma_2) : [0, 1] \rightarrow G \backslash (\mathcal{B} \times \mathcal{B})$$

satisfying the boundary equation

$$(\gamma_1(0), \gamma_2(0)) = (\eta(\gamma_2(1)), \eta(\gamma_1(1)))$$

An alternative way to state the theorem is to say that \mathcal{St}^η/G is the second component of the loop space of the quotient of $G \backslash (\mathcal{B} \times \mathcal{B})$ by the $\mathbf{Z}/2\mathbf{Z}$ -action

$$(B_1, B_2) \mapsto (\eta(B_2), \eta(B_1)).$$

(To be precise, to recover \mathcal{St}^η/G , one must also equip such loops with a trivialization of their associated $\mathbf{Z}/2\mathbf{Z}$ -torsor.) From this perspective, we see that there is a canonical S^1 -action of loop rotation on \mathcal{St}^η/G .

For any $\alpha \in H$, we write \mathcal{St}_α^η for the inverse image in \mathcal{St}^η of the formal neighborhood of α under the canonical projection. We can now state the loop interpretation of the categories appearing as spectral parameters in Soergel's conjecture.

Theorem 2.8. *For any $\alpha \in H$ there is a canonical quasi-equivalence of periodic dg derived categories*

$$L_{\text{coh}}(\mathcal{St}_\alpha^\eta/G)_{\text{per}}^{S^1} \simeq \bigoplus_{\sigma \in \Sigma(\eta, \alpha)} \mathcal{D}_{K_{\alpha, \sigma}}(\mathcal{B}_\alpha) \otimes_{\mathbf{C}} \mathbf{C}[u, u^{-1}]$$

where the right hand side is the periodic version of Soergel's category of Langlands parameters.

In parallel with the complex case, the adjoint quotient $\mathcal{St}_\alpha^\eta/G$ is not the Hochschild space of the union of the geometric parameter spaces $K_{\alpha, \sigma} \backslash \mathcal{B}_\alpha$ appearing in the right hand side of the theorem. Rather the Hochschild space of the union canonically sits inside of $\mathcal{St}_\alpha^\eta/G$, and the restriction of coherent sheaves along this embedding gives an equivalence.

The above theorem gives a description of \mathcal{D} -modules on the geometric parameter spaces $K_{\alpha, \sigma} \backslash \mathcal{B}_\alpha$ as part of a nice family with respect to the parameter α . Namely, these categories can be recovered from the loop spaces of $K_{\alpha, \sigma} \backslash \mathcal{B}_\alpha$, and the loop spaces in turn fit into the nice family formed by the equivariant Langlands parameter space $\mathcal{St}_\alpha^\eta/G$. In this setting, it is crucial that we sought such a uniform picture in the realm of loop spaces rather than cotangent bundles.

3. APPLICATIONS

In this section we outline how the results of this paper fit into our ongoing project [BN1, BN2, BN3] to apply ideas from the geometric Langlands program to representation theory of real groups, specifically to Vogan duality and Soergel's conjecture, which give refined forms of the local Langlands program over the reals.

In broad outline, we relate the local geometric Langlands program to the real local Langlands program using two principles, S^1 -equivariantization and geometric base change. The local geometric Langlands program describes module categories over loop groups and their Hecke algebras in terms of coherent sheaves on spaces of Langlands parameters. Equivariant localization for loop rotation relates the loop group (and affine Hecke algebras) to the group G (and finite Hecke algebras), and coherent sheaves on Langlands parameters to \mathcal{D} -modules on flag varieties of the dual group. This latter step is the role of the current paper. Thus representation theory of G (the complex local Langlands program) arises as the S^1 -equivariantization (or "string

states”) of representation theory of LG . On the other hand, the geometric base change conjecture relates the geometric Langlands programs over complex and real curves. The result is the real local Langlands classification identifying categories of representations of real forms of G through their local Langlands parameters, which are \mathcal{D} -modules on dual flag varieties [ABV, So].

In Section 3.1 we explain the application of the localization principle in the complex case and the resulting duality for finite Hecke categories [BN2]. In Section 3.2 we describe a real form of the geometric Langlands conjecture on \mathbf{P}^1 and its application to Vogan duality [BN1]. Finally, in Section 3.3 we introduce the geometric base change conjecture, and explains how it implies a strong form of Soergel’s conjecture [BN3].

3.1. Ramified geometric Langlands on \mathbf{P}^1 . Fix a complex reductive group G with Langlands dual group G^\vee . When referring to objects associated to G^\vee , we will often adjoin the superscript $^\vee$ to our usual notation without further comment. So for example, as usual \mathcal{B} will denote the flag variety of G , and \mathcal{B}^\vee the flag variety of G^\vee .

The equivariant Steinberg variety St^\vee/G^\vee admits a natural interpretation as the space of G^\vee -local systems on \mathbf{P}^1 with parabolic structure (simple poles and compatible flags) at the points $0, \infty \in \mathbf{P}^1$. In particular, the map to the adjoint quotient G^\vee/G^\vee gives the monodromy of a local system around the equator. Under this interpretation, the S^1 -action on St^\vee/G^\vee by loop rotation coincides with the \mathbf{C}^\times -action induced by the standard rotation of \mathbf{P}^1 fixing $0, \infty$. Note that the \mathbf{C}^\times -action reduces to an S^1 -action since it is infinitesimally trivialized by the local system structure.

Next consider the moduli stack \mathbf{Bun} of G -bundles on \mathbf{P}^1 equipped with flags at the points $0, \infty \in \mathbf{P}^1$. The geometric Langlands program predicts an intimate connection between the derived category of coherent sheaves $L_{\text{coh}}(St^\vee/G^\vee)$ and the derived category of monodromic \mathcal{D} -modules $\mathcal{D}(\mathbf{Bun})_{\text{mon}}$. By Bezrukavnikov’s work, both of the above categories are module categories for the affine Hecke category. On the one hand, $L_{\text{coh}}(St^\vee/G^\vee)$ is nothing more than the regular module category. On the other hand, in [BN2], we show that $\mathcal{D}(\mathbf{Bun})_{\text{mon}}$ is the dual module category (in a precise sense which would take some space to spell out).

One can interpret the duality of affine Hecke module categories

$$\mathcal{D}(\mathbf{Bun})_{\text{mon}} \leftrightarrow L_{\text{coh}}(St^\vee/G^\vee)$$

as the *tamely ramified geometric Langlands correspondence* on \mathbf{P}^1 . A key property of the duality is that it respects automorphisms of the curve \mathbf{P}^1 fixing the points $0, \infty \in \mathbf{P}^1$. Namely, the S^1 -action on $L_{\text{coh}}(St^\vee/G^\vee)$ by loop rotation is transported to the \mathbf{C}^\times -action on $\mathcal{D}(\mathbf{Bun})_{\text{mon}}$ induced by the standard rotation of \mathbf{P}^1 . Note that here as well the \mathbf{C}^\times -action also reduces to an S^1 -action since it is infinitesimally trivialized by the \mathcal{D} -module structure.

Consider for a moment the moduli stack $\mathbf{Bun}_{\text{mon}}$ of G -bundles on \mathbf{P}^1 with unipotent level structure at the points $0, \infty \in \mathbf{P}^1$. One can interpret objects of $\mathcal{D}(\mathbf{Bun})_{\text{mon}}$ as \mathcal{D} -modules on $\mathbf{Bun}_{\text{mon}}$ that are constructible along the fibers of the projection $\mathbf{Bun}_{\text{mon}} \rightarrow \mathbf{Bun}$. A key observation is that the \mathbf{C}^\times -action on $\mathbf{Bun}_{\text{mon}}$ by the standard rotation of \mathbf{P}^1 does *not* reduce to an S^1 -action since its orbits contain nontrivial moduli of objects. Rather the action reveals important structure as summarized in the following statement.

Observation 3.1. The fixed points of the natural \mathbf{C}^\times -action on $\mathbf{Bun}_{\text{mon}}$ are precisely the open locus $\mathbf{Bun}_{\text{mon}}^\circ$ of trivial G -bundles with parabolic structure.

As explained in [BN2], the general principle of S^1 -equivariant localization applies directly to the derived category of \mathcal{D} -modules on \mathbf{Bun}_{mon} . The localization of the category of \mathbf{C}^\times -equivariant objects is equivalent to the periodic version of the derived category of \mathcal{D} -modules on the fixed points \mathbf{Bun}_{mon}° . By Observation 3.1, the fixed points can be identified with the quotient of a product of monodromic flag varieties

$$\mathbf{Bun}_{mon}^\circ \simeq G \backslash (\mathcal{B}_{mon} \times \mathcal{B}_{mon})$$

Here by the monodromic flag variety \mathcal{B}_{mon} , we mean the moduli of a Borel subgroup $B \subset G$, together with an element $h \in B/U$.

Combining the above discussion with the results of this paper, we can summarize the situation in the following schematic diagram.

$$\begin{array}{ccc}
 \text{Automorphic side} & & \text{Spectral side} \\
 \mathcal{D}_G(\mathcal{B} \times \mathcal{B})_{mon} & \xrightarrow{\text{complex local Langlands equivalence}} & \mathcal{D}_{G^\vee}(\mathcal{B}^\vee \times \mathcal{B}^\vee) \\
 \text{Observation 3.1} \downarrow & & \downarrow \text{Theorem 2.6} \\
 \mathcal{D}(\mathbf{Bun}^\circ)_{mon} & & L_{\text{coh}}(\mathcal{S}t^\vee/G^\vee)^{S^1} \\
 \mathbf{C}^\times\text{-fixed points} \uparrow & & \uparrow S^1\text{-fixed points} \\
 \mathcal{D}(\mathbf{Bun})_{mon} & \xrightarrow{\text{Ramified geometric Langlands}} & L_{\text{coh}}(\mathcal{S}t^\vee/G^\vee)
 \end{array}$$

The equivalence on the bottom row, as discussed above, is a form of the tamely ramified local geometric Langlands theorem of Bezrukavnikov, i.e. Langlands duality for affine Hecke categories. The equivalence of the top row is a Langlands duality for finite Hecke categories, and a form of the complex local Langlands classification. The existence of such an equivalence is a theorem of Soergel [So] whose proof is based on the Koszul duality theorem of Beilinson, Ginzburg, and Soergel [BGS]. (Concretely, it derives from a calculation of Ext groups of generators on both sides.) Via Beilinson-Bernstein localization, the left hand side is the category of Harish-Chandra bimodules with trivial generalized infinitesimal character, and the equivalence is the complex case of Soergel's general conjecture on Langlands parametrization of categories of Harish-Chandra modules for real groups. The arguments we have sketched here (and develop in detail in [BN2]) provide a *canonical* construction and characterization of this equivalence as well as a conceptual explanation for its existence. These results can also be viewed as a tamely ramified and conceptual version of the S^1 -equivariant geometric Satake correspondence of Bezrukavnikov and Finkelberg [BeFi], which relates the \mathbb{G}_m -equivariant version of the derived Satake category to Harish-Chandra bimodules for the dual group.

3.2. Geometric Langlands for real groups. Next we introduce real forms of G into the geometric Langlands correspondence. Fix once and for all a quasi-split conjugation θ of G , and let η be the combinatorially corresponding algebraic involution of G^\vee .

Consider the antipodal conjugation of \mathbf{P}^1 , and note that it exchanges the points $0, \infty \in \mathbf{P}^1$. Thus together with the conjugation θ , it provides a real form \mathbf{Bun}_θ of the moduli stack \mathbf{Bun} . Similarly, we have the corresponding real form $\mathbf{Bun}_{\theta, mon}$ of the monodromic moduli stack \mathbf{Bun}_{mon} . Observe that since $0, \infty \in \mathbf{P}^1$ are exchanged by the antipodal conjugation, the natural projection $\mathbf{Bun}_{\theta, mon} \rightarrow \mathbf{Bun}_\theta$ is a torsor for a single copy of the universal Cartan H .

The restriction of the standard \mathbf{C}^\times -action on \mathbf{P}^1 to the unitary circle $U(1) \subset \mathbf{C}^\times$ preserves the antipodal conjugation. Thus we have the induced $U(1)$ -action on the above moduli stacks. The orbits of the action on \mathbf{Bun}_θ are discrete, but those of the action on \mathbf{Bun}_θ have moduli.

Observation 3.2. The fixed points of the natural $U(1)$ -action on $\mathbf{Bun}_{\theta, \text{mon}}$ are precisely the open locus $\mathbf{Bun}_{\theta, \text{mon}}^\circ$ where the underlying G -bundle is trivial.

Fix $\alpha \in H^\vee$, and consider the derived category $\text{Sh}(\mathbf{Bun}_\theta)_\alpha$ of monodromic constructible sheaves on \mathbf{Bun}_θ with monodromicity α . As explained in [BN1], the general principle of S^1 -equivariant localization applies in this setting: the localization of the category of $U(1)$ -equivariant objects of $\text{Sh}(\mathbf{Bun}_\theta)_\alpha$ is equivalent to the differential $\mathbf{Z}/2\mathbf{Z}$ -graded version of the derived category $\text{Sh}(\mathbf{Bun}_\theta^\circ)_\alpha$ of monodromic constructible sheaves on the fixed points. Using Observation 3.1, one can show that the fixed points are a union of real quotients of monodromic flag varieties

$$\mathbf{Bun}_{\theta, \text{mon}}^\circ \simeq \coprod_{\theta' \in \Theta(\eta)} G_{\mathbf{R}, \theta'} \backslash \mathcal{B}_{\text{mon}}.$$

Here the index set $\Theta(\eta)$ is precisely the one arising in Vogan duality and Soergel's conjecture. Now there is an analytic version of Beilinson-Bernstein localization due to Kashiwara-Schmid [KS] that localizes representations of a real form $G_{\mathbf{R}, \theta'}$ on the corresponding real quotient of the monodromic flag variety $G_{\mathbf{R}, \theta'} \backslash \mathcal{B}_{\text{mon}}$. Thus for $\lambda \in \mathfrak{h}^\vee$ with $\alpha = \exp(\lambda)$, we have an identification of derived categories

$$\text{Sh}(\mathbf{Bun}_\theta^\circ)_\alpha \simeq \bigoplus_{\theta' \in \Theta(\eta)} \mathcal{D}(\mathcal{HC}_{\theta', [\lambda]}).$$

Here the right hand side (or rather its pro-completion) is the derived category of Harish-Chandra modules appearing in Soergel's conjecture.

Now we can summarize our program to understand Soergel's conjecture in the following schematic diagram. The left hand automorphic column has been sketched in the preceding discussion. The right hand spectral column follows from the results of this paper as described in the overview. Finally, the bottom horizontal arrow is a conjectural real geometric Langlands correspondence relating module categories for the affine Hecke category. (Some intrinsic motivation for the form of this statement will be given in the subsequent section.)

<u>Automorphic side</u>		<u>Spectral side</u>
$\bigoplus_{\theta' \in \Theta(\eta)} \mathcal{D}(\mathcal{HC}_{\theta', [\lambda]})$	$\xrightarrow{\text{Soergel conjecture}}$	$\bigoplus_{\sigma \in \Sigma(\eta, \alpha)} \mathcal{D}_{K_{\alpha, \sigma}^\vee}(\mathcal{B}_\alpha^\vee)$
$\downarrow \text{Localization of [KS]}$		$\downarrow \text{Theorem 2.8}$
$\text{Sh}(\mathbf{Bun}_\theta^\circ)_\alpha$		$L_{\text{coh}}(\mathcal{S}t_\alpha^{\vee, \eta}/G^\vee)^{S^1}$
$\uparrow U(1)\text{-fixed points}$		$\uparrow S^1\text{-fixed points}$
$\text{Sh}(\mathbf{Bun}_\theta)_\alpha$	$\xrightarrow{\text{Real geometric Langlands conjecture}}$	$L_{\text{coh}}(\mathcal{S}t_\alpha^{\vee, \eta}/G^\vee)$

By our other results, a construction of the real geometric Langlands correspondence would resolve Soergel's conjecture. In the next section, we will discuss how such a correspondence follows from a conjectural *geometric base change* principle. This is something we can currently verify holds on the level of Grothendieck groups. Coupling it with the Kazhdan-Lusztig theorem on the affine Hecke algebra, we obtain the following affine version of Vogan duality (or real Kazhdan-Lusztig theorem).

Theorem 3.3 ([BN1]). (*Affine Vogan Duality*) For any $\alpha \in H^\vee$, there is a canonical duality of modules for the affine Weyl group

$$K(\mathrm{Sh}(\mathbf{Bun}_\theta)_\alpha) \leftrightarrow K(L_{\mathrm{coh}}(\mathcal{S}t_\alpha^{\vee,\eta}/G^\vee)).$$

It is worth emphasizing that the proof of this theorem does not depend on difficult categorical statements such as found in the work of Bezrukavnikov, but only the easier original K -theoretic Kazhdan-Lusztig theorem.

As a corollary, we obtain a canonical and conceptual new proof of Vogan duality.

Corollary 3.4. (*Vogan Duality*) Vogan's character duality isomorphism

$$\bigoplus_{\theta' \in \Theta(\eta)} K(\mathcal{HC}_{\theta', [\lambda]}) \leftrightarrow \bigoplus_{\sigma \in \Sigma(\eta, \alpha)} K(\mathcal{D}_{K_{\alpha, \sigma}}(\mathcal{B}_\alpha))$$

follows by applying S^1 -equivariant localization to Theorem 3.3.

3.3. Geometric base change. In this section, we sketch results from [BN3] explaining how the principle of base change from the Langlands program manifests in the geometric setting. In particular, a geometric form of base change provides a proof of Theorem 3.3, and in particular, a conceptual proof of Vogan duality. If geometric base change can be verified on \mathbf{P}^1 , it will also provide a proof of Soergel's conjecture. More generally, when it can be verified, it reduces the real geometric Langlands correspondence to the usual complex version.

The following schematic diagram summarizes how base change fits into our preceding discussion. The horizontal arrows have been discussed in the two preceding sections. Our aim here is to explain the vertical arrows. The right hand spectral base change is a result from [BN3]. The left hand automorphic base change is a conjecture on the categorical level, and a theorem on the level of Grothendieck groups [BN1]. To simplify what is a very general discussion, we will suppress any further mention of monodromic parameters.

$$\begin{array}{ccc} \text{Automorphic side} & & \text{Spectral side} \\ \mathrm{Sh}(\mathbf{Bun}_\theta) & \xrightarrow{\text{Real geometric Langlands conjecture}} & L_{\mathrm{coh}}(\mathcal{S}t^{\vee,\eta}/G^\vee) \\ \text{Conjectural base change} \downarrow & & \downarrow \text{Spectral base change} \\ \mathrm{Sh}(\mathbf{Bun}) & \xrightarrow{\text{Ramified geometric Langlands}} & L_{\mathrm{coh}}(\mathcal{S}t^\vee/G^\vee) \end{array}$$

Recall that the equivariant Steinberg variety $\mathcal{S}t^\vee/G^\vee$ admits a natural interpretation as the space of G^\vee -local systems on \mathbf{P}^1 with parabolic structure (simple poles and compatible flags) at the points $0, \infty \in \mathbf{P}^1$. Likewise, the equivariant Langlands parameter space $\mathcal{S}t^{\vee,\eta}/G^\vee$ admits a natural interpretation as the space of G^\vee -local systems on \mathbf{P}^1 with parabolic structure at the points $0, \infty \in \mathbf{P}^1$, and an η -twisted $\mathbf{Z}/2\mathbf{Z}$ -equivariance under the antipodal involution of \mathbf{P}^1 . It is often illuminating to think of such an object as an η -twisted local system on the quotient space $\mathbf{RP}^1 = \mathbf{P}^1/\mathbf{Z}/2\mathbf{Z}$ with a parabolic structure at the point $0 = \infty \in \mathbf{RP}^1$. From this perspective, it is clear from Tannakian principles why we conjecture that $L_{\mathrm{coh}}(\mathcal{S}t^{\vee,\eta}/G^\vee)$ should provide the spectral description for $\mathrm{Sh}(\mathbf{Bun}_\theta)$ under a real geometric Langland correspondence. In what follows, we will also give a motivation for this answer using the principle of base change.

Consider the general situation of a covering map of curves $\hat{C} \rightarrow C$ with Galois group Γ so that $C = \hat{C}/\Gamma$. Consider the stack of G^\vee -connections $\mathrm{Conn}_{G^\vee}(\hat{C})$, and its derived category of coherent sheaves $L_{\mathrm{coh}}(\mathrm{Conn}_{G^\vee}(\hat{C}))$. As explained in [BN3], a simple spectral base change argument recovers $L_{\mathrm{coh}}(\mathrm{Conn}_{G^\vee}(C))$ from natural operations on $L_{\mathrm{coh}}(\mathrm{Conn}_{G^\vee}(\hat{C}))$. Namely,

for each point of \hat{C} , we have a tautological action of the tensor category $\text{Rep}(G^\vee)$ of representations on $L_{\text{coh}}(\text{Conn}_{G^\vee}(\hat{C}))$, and imposing that the action is identified for Γ -related points is precisely the correct descent data. One can go further and given an action of Γ on G^\vee (or more generally, a compatible action on a group-scheme over \hat{C}) extend this picture to obtain a Galois-twisted version. It is also worth remarking that the construction is compatible with respect to automorphisms of the curves.

As a special case, taking $\hat{C} = \mathbf{P}^1$, and $C = \mathbf{RP}^1$, with $\Gamma = \mathbf{Z}/2\mathbf{Z}$ acting on \mathbf{P}^1 via the antipodal map, and on G^\vee via the involution η , we see that $L_{\text{coh}}(\mathcal{S}t^{\vee, \eta}/G^\vee)$ can be recovered from $L_{\text{coh}}(\mathcal{S}t^\vee/G^\vee)$ by spectral base change. Thus combining this with the results of the current paper on S^1 -equivariant localization, we see that the derived category of equivariant \mathcal{D} -modules on the geometric parameter spaces that appears in Soergel's conjecture can be obtained from $L_{\text{coh}}(\mathcal{S}t^\vee/G^\vee)$ by entirely formal categorical considerations.

In [BN3], we formulate and study the geometric version of the base change principle in the automorphic setting. In general, given a covering map of curves $\hat{C} \rightarrow C$, we arrive at a conjecture that relates categories of \mathcal{D} -modules on the moduli space $\text{Bun}_G(\hat{C})$ of G -bundles to \mathcal{D} -bundles on the moduli space $\text{Bun}_G(C)$. As in the spectral setting, the construction is purely categorical involving only the natural Hecke operators of the theory.

In the special case when $\hat{C} = \mathbf{P}^1$, and $C = \mathbf{RP}^1$, with $\Gamma = \mathbf{Z}/2\mathbf{Z}$ acting on \mathbf{P}^1 via the antipodal map, and on G via the conjugation θ , we arrive at a conjectural way to recover $\text{Sh}(\mathbf{Bun}_\theta)$ directly from $\text{Sh}(\mathbf{Bun})$. It is worth emphasizing that this statement together with the results of the current paper says that we should be able to see all of the complicated categorical structures in the representation theory of real groups directly from the complex case by abstract nonsense. In fact, in this special case, we are able to verify automorphic base change on the level of Grothendieck groups, hence we already know a large part of the combinatorics satisfies this principle. The argument is the primary ingredient in the proof of Theorem 3.3, and packages all of the combinatorics in Vogan duality into a concise conceptual framework.

A proof of automorphic base change at the categorical level, combined with the work of Bezrukavnikov in the complex case and the results of the current paper, would provide a proof of Soergel's conjecture. In fact, this line of argument gives an improved formulation since all of the steps are canonical, and compatible with Hecke actions. By contrast, Soergel conjectures an equivalence via the existence of generators on both sides with isomorphic endomorphism algebras.

Theorem 3.5 ([BN3]). *A canonical, Hecke-equivariant form of Soergel's conjecture follows from the geometric base change conjecture for the antipodal map on \mathbf{P}^1 .*

4. LOOP AND HOCHSCHILD SPACES

In this paper, all rings are assumed to be commutative, unital and over a field k of characteristic 0.

The theory of derived stacks provides a useful language to discuss the objects of representation theory that interest us. For the reader's convenience, we have provided a brief Appendix summarizing some of the basic motivation and terminology from the theory of quasicategories and derived stacks. For derived stacks, we refer the reader to Toën's extremely useful survey [To2], and to the papers of Toën-Vezzosi [ToVe1, ToVe2] and Lurie [L1, L3, L4, L5]. In particular, we were introduced to derived loop spaces by [To2]. We will not need any deep statements from this theory, only the formalism that allows us to perform basic constructions

and to obtain well-behaved categories of sheaves. We will work in the context of quasicategories [Jo, Ber] (or ∞ -categories in the language of [L2]). As explained in the Appendix (see the survey [Ber]), this is one of many equivalent categorical contexts for homotopical algebra and geometry which lie in between the coarse world of homotopy categories and the fine world of model categories. In particular, the Dwyer-Kan simplicial localization of a model category provides a primary source of quasicategories.

In this section, we collect the definitions and basic properties of the loop space $\mathcal{L}X$ of a derived stack. We then focus on the space of small loops or Hochschild space $\mathcal{H}X$ obtained by formally completing $\mathcal{L}X$ along the constant loops X . One motivation for introducing small loops is that they are local in X in a suitable sense. With our intended applications in mind, we content ourselves with the concrete situation when X is a smooth Artin stack (though the discussion surely holds in far greater generality). We show that $\mathcal{H}X$ is isomorphic to the formal completion of the odd tangent bundle along its zero section. Passing to functions gives an isomorphism of the Hochschild homology and de Rham algebra of X . One can view this as a version of the Hochschild-Kostant-Rosenberg theorem in the context of stacks. Although not needed in what follows, we also review other well-known structures such as the Atiyah bracket on Hochschild cohomology and its interpretation in this context.

4.1. The loop space $\mathcal{L}X$. Let \mathcal{S} denote the quasicategory of simplicial sets, or equivalently (compactly generated Hausdorff) topological spaces. Let \mathcal{SCA}_k denote the quasicategory of simplicial commutative unital k -algebras, or equivalently connective commutative differential graded k -algebras. An object of the quasicategory of derived stacks over k is a functor

$$X : \mathcal{SCA}_k \rightarrow \mathcal{S}$$

which is a sheaf in the étale topology.

Some natural classes of derived stacks are provided by derived schemes (in particular, representable functors given by affine derived schemes), Artin stacks, and topological spaces. For the latter, any compactly generated Hausdorff topological space K defines a derived stack given by the sheafification of the constant functor

$$K : \mathcal{SCA}_k \rightarrow \mathcal{S} \quad K(R) = K.$$

To connect with other combinatorial models, it is often convenient to choose a simplicial presentation of K (for example, that given by singular chains) and consider K as a functor to simplicial sets. Of course, any two simplicial presentations lead to equivalent stacks.

Given derived stacks K, X , morphisms of sheaves form a derived mapping stack $\mathbf{R}\mathrm{Hom}(K, X)$. When K is a topological space, we can think of $\mathbf{R}\mathrm{Hom}(K, X)$ as a collection of equations imposed on copies of X . One can check that for K a finite simplicial set and X a derived Artin stack, the derived mapping stack $\mathbf{R}\mathrm{Hom}(K, X)$ is also a derived Artin stack. (The reader could consult the Appendix for a discussion on what it means for a derived stack to be Artin.)

In this paper, we will focus on the locally constant stack given by the circle $K = S^1$, which is identified with the classifying space $B\mathbf{Z}$. In this case, we refer to the corresponding derived mapping stack $\mathbf{R}\mathrm{Hom}(S^1, X)$ as the loop space of X and denote it by $\mathcal{L}X$. Roughly speaking, we take a copy of X and impose the equation that every point of X must be equal to itself.

To make this concrete, we can choose a simplicial presentation of S^1 . For example, a particularly small presentation of S^1 as a simplicial set has two 0-simplices, two nontrivial 1-simplices, and no nontrivial higher simplices. This leads to the usual model of the loop space as the derived fiber product

$$\mathcal{L}X \simeq X \times_{X \times X}^{\mathbf{R}} X$$

along the diagonal maps. Thus the pushforward to X of functions on $\mathcal{L}X$ is given by the derived tensor product

$$\mathcal{O}_X \otimes_{\mathcal{O}_{X \times X}}^{\mathbf{L}} \mathcal{O}_X.$$

When X has affine diagonal, so that $\mathcal{L}X$ is affine over X , we can think of $\mathcal{L}X$ as the spectrum over X of the derived tensor product.

Two examples are useful to keep in mind. When X is an ordinary affine scheme, the component ring $\pi_0(\mathcal{O}_{\mathcal{L}X})$ is nothing more than \mathcal{O}_X , thus the underlying ordinary scheme of $\mathcal{L}X$ is simply X itself, and so $\mathcal{L}X$ is a purely derived enhancement. On the other hand, if $X = BG$ is the classifying space of an algebraic group G , it is easy to see that $\mathcal{L}X = G/G$ is the adjoint quotient stack with trivial derived structure.

Of the many interesting structures on $\mathcal{L}X$, we will concentrate on the S^1 -action given by loop rotation

$$S^1 \times \mathcal{L}X \rightarrow \mathcal{L}X.$$

Connes' theory of cyclic objects [Co] provides a convenient algebraization of S^1 and more generally of S^1 -spaces (see [J] for an application to free loop spaces and [Lo] for a detailed exposition). Consider S^1 as the unit circle in the complex plane, and let $\mathbf{Z}_{n+1} = \{z \in S^1 \mid z^{n+1} = 1\}$ denote the $(n+1)$ st roots of unity. Connes' cyclic category $\mathbf{\Lambda}$ is the category with objects finite ordered sets $\mathbf{n} = \{0, 1, \dots, n\}$, and morphisms homotopy classes of continuous, order preserving, degree one maps of pairs

$$\mathbf{\Lambda}(\mathbf{n}, \mathbf{m}) = [s : (S^1, \mathbf{Z}_{n+1}) \rightarrow (S^1, \mathbf{Z}_{m+1})].$$

Here a map $s : S^1 \rightarrow S^1$ is said to be order preserving if any lift $\tilde{s} : \mathbf{R} \rightarrow \mathbf{R}$ is non-decreasing. The geometric realization of the simplicial set

$$\Lambda = \mathbf{\Lambda}(-, [0])$$

is homeomorphic to S^1 .

This presentation of S^1 leads to the familiar model of $\mathcal{L}X$ as a cocyclic space with n -cosimplices given by the products X^{n+1} (with the usual diagonal and projection structure maps). The pushforward to $X \times X$ of functions on $\mathcal{L}X$ is given by the usual cyclic complex of Hochschild chains

$$\mathcal{C}^{-\bullet}(\mathcal{O}_X) = \mathcal{B}^{-\bullet}(\mathcal{O}_X) \otimes_{\mathcal{O}_{X \times X}} \mathcal{O}_X$$

where the bar resolution of \mathcal{O}_X is given by

$$\mathcal{B}^{-\bullet}(\mathcal{O}_X) = \cdots \rightarrow \mathcal{B}^{-2}(\mathcal{O}_X) \xrightarrow{\partial} \mathcal{B}^{-1}(\mathcal{O}_X) \xrightarrow{\partial} \mathcal{B}^0(\mathcal{O}_X)$$

with terms

$$\mathcal{B}^{-q}(\mathcal{O}_X) = \mathcal{O}_{X^{(q+2)}} = \mathcal{O}_X^{\otimes(q+2)} = \mathcal{O}_X \otimes \cdots \otimes \mathcal{O}_X,$$

with $\mathcal{O}_{X \times X}$ -module structure

$$(a_\ell \otimes a_r) \cdot (r_0 \otimes \cdots \otimes r_{q+1}) = a_\ell r_0 \otimes \cdots \otimes r_{q+1} a_r,$$

and differential

$$\partial(r_0 \otimes \cdots \otimes r_{q+1}) = \sum_{i=0}^q (-1)^i r_0 \otimes \cdots \otimes r_i r_{i+2} \otimes \cdots \otimes r_{q+1}.$$

4.2. The Hochschild space $\mathcal{H}X$. Let X denote a derived stack and let $\mathcal{L}X$ denote its loop space. Consider the canonical projection $\mathcal{L}X \rightarrow X$ given by the evaluation of loops at the base point. One might naively hope that the loop space functor satisfies some form of descent with respect to maps to X . But as in more traditional contexts, it is impossible to realize all loops by gluing together local loops. We can find a local version of loops if we restrict from all loops to “small loops”.

To make this precise, consider the canonical map $X \rightarrow \mathcal{L}X$ that sends a point to the constant loop at that point.

Definition 4.1. The *Hochschild space* (or small loop space) $\mathcal{H}X$ of a derived stack X is the derived stack $\mathcal{H}X = \widehat{\mathcal{L}X}_X$ obtained as the formal completion of the loop space $\mathcal{L}X$ along the constant loops $X \rightarrow \mathcal{L}X$.

Note that since the constant loops $X \rightarrow \mathcal{L}X$ are preserved by loop rotation, the S^1 -action on $\mathcal{L}X$ descends to an action on the Hochschild space $\mathcal{H}X$.

To return to our two previous examples, when X is an ordinary affine scheme, the Hochschild space $\mathcal{H}X$ coincides with the loop space $\mathcal{L}X$. On the other hand, for $X = BG$ the classifying space of an algebraic group G , the Hochschild space $\mathcal{H}BG = \widehat{G}/G$ is the stack quotient of the formal group \widehat{G} by the adjoint action of G .

For the reader’s convenience, let us spell out what it means to take the formal completion in the language of functors of points. We will give an interpretation in which we separate the derived aspect of the situation from the formal. For $R \in \mathcal{SCA}_k$, let $S = \text{Spec } R$ be the affine derived scheme given by the representable functor

$$\text{Hom}(R, -) : \mathcal{SCA}_k \rightarrow \mathcal{S}.$$

The components $\pi_0(R)$ form an ordinary discrete commutative algebra and we have a canonical projection $R \rightarrow \pi_0(R)$. Thus the ordinary affine scheme $S_0 = \text{Spec } \pi_0(R)$ comes equipped with a canonical map $S_0 \rightarrow S$ of affine derived schemes. In this way, we may think of S as a kind of “derived thickening” of S_0 .

Consider the nilradical $\mathcal{N} \subset \pi_0(R)$ and the corresponding reduced affine scheme $S_{0,r} = \text{Spec } \pi_0(R)/\mathcal{N}$. Via the canonical map $S_{0,r} \rightarrow S_0$, we may think of S_0 as a “formal thickening” of $S_{0,r}$. Now given a map of derived stacks $X \rightarrow Y$, the formal completion \widehat{Y}_X of Y along X assigns to the affine derived scheme S the space of homotopy commutative diagrams

$$\begin{array}{ccc} S & \rightarrow & Y \\ \uparrow & & \uparrow \\ S_{0,r} & \rightarrow & X \end{array}$$

To be precise, maps from the test object S into the formal completion \widehat{Y}_X are given by the homotopy fiber product

$$\text{Hom}(S, \widehat{Y}_X) = \text{Hom}(S, Y) \times_{\text{Hom}(S_{0,r}, Y)}^{\mathbb{R}} \text{Hom}(S_{0,r}, X).$$

4.3. Case of Artin stacks. In what follows, we will restrict our study of the loop space $\mathcal{L}X$ and Hochschild space $\mathcal{H}X$ to the situation where X itself is a smooth Artin stack with affine diagonal. Not only will this assumption simplify the discussion, but our intended applications in representation theory fit into this context. Our need to consider derived stacks arises from the fact that they appear as a result of the loop space construction.

Let X be a smooth Artin stack with affine diagonal. Consider a presentation of X with objects and morphisms given by smooth schemes X_0 and X_1 respectively, and groupoid structure

maps denoted as follows

$$\ell, r : X_1 \rightarrow X_0 \quad e : X_0 \rightarrow X_1 \quad m : X_1 \times_{X_0} X_1 \rightarrow X_1 \quad i : X_1 \rightarrow X_1.$$

Consider the pushforward to X of the structure sheaf $\mathcal{O}_{\mathcal{L}X}$. By construction, it can be realized as the descent from X_0 of the derived tensor product

$$\mathcal{O}_{X_1} \otimes_{X_0 \times X_0}^{\mathbf{L}} \mathcal{O}_{\Delta_{X_0}}$$

where X_1 maps to $X_0 \times X_0$ by the product $\ell \times r$, and Δ_{X_0} denotes the diagonal of $X_0 \times X_0$. In other words, loops are thought of as pairs of a 1-simplex and a 0-simplex such that the two ends of the 1-simplex are glued to the 0-simplex.

The first result we will need is a characterization when the loop space $\mathcal{L}X$ has trivial derived structure.

Proposition 4.2. *The loop space $\mathcal{L}X$ has trivial derived structure if and only if the isomorphism classes of objects X are discrete.*

Proof. It is convenient to rewrite the pushforward to X of the structure sheaf $\mathcal{O}_{\mathcal{L}X}$ as the descent of the derived tensor product

$$\mathcal{O}_{X_1} \otimes_{X_0 \times X_1}^{\mathbf{L}} \mathcal{O}_{X_1}$$

where X_1 maps to $X_0 \times X_1$ by the product maps $\ell \times \text{id}_{X_1}$ and $r \times \text{id}_{X_1}$. This can be viewed as the structure sheaf of the derived intersection of the subschemes $\Gamma_\ell, \Gamma_r \subset X_0 \times X_1$ given by the graphs of ℓ, r respectively. Here loops are thought of as pairs of 1-simplices that are equal and such that the left end of the first is glued to the right end of the second.

Now our assertion will follow from a simple dimension count. Let n_0 and n_1 be the dimensions of X_0 and X_1 respectively. On the one hand, the expected dimension of the intersection $\Gamma_\ell \cap \Gamma_r$ inside of $X_0 \times X_1$ is given by $n_1 + n_1 - (n_0 + n_1) = n_1 - n_0$. On the other hand, each isomorphism class of objects of X contributes a subscheme of precisely dimension $n_1 - n_0$ to the intersection. Thus the intersection has the expected dimension if and only if there is no nontrivial moduli of isomorphism classes of objects. \square

The next result we will need is an identification of the Hochschild space $\mathcal{H}X$ with the completed odd tangent bundle.

Let \mathbb{T}_{X_0} denote the tangent sheaf of X_0 , and let \mathfrak{g}_{X_1} denote the Lie algebroid on X_0 associated to the groupoid. Recall that the tangent complex of X is the descent from X_0 of the complex built out of the action map

$$\alpha : \mathfrak{g}_{X_1}[1] \rightarrow \mathbb{T}_{X_0}.$$

By definition, the odd tangent bundle of X is the descent from X_0 of the spectrum of the symmetric algebra of the shifted cotangent complex

$$\mathbb{T}_X[-1] = \text{Spec}(\text{Sym}_{\mathcal{O}_X}^{\bullet}(\Omega_X^1[1] \xrightarrow{\alpha^*} \mathfrak{g}_{X_1}^*)).$$

We write $\widehat{\mathbb{T}}_X[-1]$ for the completion of $\mathbb{T}_X[-1]$ along the base X and call it the completed odd tangent bundle.

Proposition 4.3. *For X a smooth Artin stack with affine diagonal, the Hochschild space $\mathcal{H}X$ is canonically isomorphic to the completed odd tangent bundle $\widehat{\mathbb{T}}_X[-1]$.*

Proof. Let \widehat{X}_{1, X_0} denote the completion of X_1 along the unit morphism $e : X_0 \rightarrow X_1$. Consider the pushforward to X of the structure sheaf $\mathcal{O}_{\mathcal{H}X}$. By construction, it can be realized as the descent from X_0 of the completed tensor product

$$\mathcal{O}_{\mathcal{H}X} = \mathcal{O}_{\widehat{X}_{1, X_0}} \widehat{\otimes}_{\mathcal{O}_{X_0 \times X_0}}^{\mathbf{L}} \mathcal{O}_{\Delta_{X_0}}.$$

Consider the formal completion of the diagonal Δ_{X_0} inside of $X_0 \times X_0$, and the associated completed Koszul resolution of the sheaf of functions $\mathcal{O}_{\Delta_{X_0}}$. Then performing the completed tensor product, we obtain a complex given by the completed symmetric product

$$\widehat{\text{Sym}}_{\mathcal{O}_X}^{\bullet}(\Omega_X^1[1] \xrightarrow{\alpha^*} \mathfrak{g}_{X_1}^*).$$

This is precisely the ring of functions on the completed odd tangent bundle $\widehat{\mathbb{T}}_X[-1]$. \square

Corollary 4.4. *The Hochschild space $\mathcal{H}X$ has trivial derived structure if and only if each irreducible component of X contains a dense isomorphism class of objects.*

Proof. The second postulate is equivalent to the injectivity of the dualized action map α^* . \square

For our purposes, the key consequence of the proposition is that the Hochschild space $\mathcal{H}X$ is a local object in the following sense. Consider the simplicial scheme X_{\bullet} with 0-simplices given by X_0 , and for $k > 0$, k -simplices given by the fiber products

$$X_k = X_1 \times_{X_0} \cdots \times_{X_0} X_1 \quad \text{with } k \text{ factors.}$$

Functions on the completed odd tangent bundle can be calculated as the limit of functions on the completed odd tangent bundles of the simplices. Thus by the proposition, functions on the Hochschild space $\mathcal{O}_{\mathcal{H}X}$ can be calculated as the limit of functions on the Hochschild spaces $\mathcal{O}_{\mathcal{H}X_k}$ of the simplices. (It is worth remarking, as pointed out by J. Lurie, that this descent for functions is not valid for the Hochschild space itself.) Observe that this presentation is compatible with the S^1 -action of loop rotation: the S^1 -action on $\mathcal{O}_{\mathcal{H}X}$ is the limit of the S^1 -actions on the terms $\mathcal{O}_{\mathcal{H}X_k}$.

By the above discussion, to understand the local geometry of $\mathcal{H}X$, it will usually suffice to study the case when X is simply a smooth affine scheme. In this case, the odd tangent bundle and the completed odd tangent bundle coincide (since both have X as their underlying schemes). Both are the spectrum of the de Rham algebra of differential forms $\Omega_X^{-\bullet}$. Furthermore, the identification of the proposition is nothing more than the Hochschild-Kostant-Rosenberg theorem. Under the functor from cyclic modules to mixed modules (see [Lo]), the S^1 -action on $\mathcal{O}_{\mathcal{H}X}$ goes over to the de Rham differential d on $\Omega_X^{-\bullet}$.

4.4. Lie structure of loop spaces. In this informal section, we mention further structures on loop spaces and place them in our current context. The discussion will not be used in the remainder of the paper.

The circle S^1 (equipped with a fixed basepoint $1 \in S^1$) has a natural comultiplication in the category of pointed spaces

$$S^1 \rightarrow S^1 \vee S^1.$$

Given any derived stack X , the loop space $\mathcal{L}X$ inherits a multiplication

$$\mathcal{L}X \times_X \mathcal{L}X \rightarrow \mathcal{L}X$$

from the comultiplication on S^1 . As usual, this multiplication is not associative but rather fits into an A_{∞} -monoid structure over X .

The circle S^1 also has a natural time-reversal automorphism fixing the base-point. Thus the loop space $\mathcal{L}X$ inherits a parametrization-reversal automorphism.

We like to summarize the situation by thinking of the loop space $\mathcal{L}X$ as a Lie group and the Hochschild space $\mathcal{H}X$ as its formal group. Taking this perspective, it is natural to ask about its Lie algebra. One can interpret this as the odd tangent bundle $\mathbb{T}_X[-1]$ equipped with its canonical Lie algebra structure

$$\mathbb{T}_X[-1] \otimes \mathbb{T}_X[-1] \rightarrow \mathbb{T}_X[-1]$$

given by the Atiyah class [Ka2]. The analogue of the enveloping algebra, or space of distributions on $\mathcal{L}X$ supported along X , is the usual Hochschild cochain complex

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times X}}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X})$$

The Yoneda product of Exts gives an A_∞ -multiplication which agrees with the convolution structure induced by the multiplication of loops. This picture was explained by Markarian [M] and furthered by Ramadoss [R1, R2] and Roberts-Willerton [RW]. From this perspective, the Hochschild-Kostant-Rosenberg isomorphism becomes the Poincaré-Birkhoff-Witt isomorphism for the Lie algebra $\mathbb{T}_X[-1]$.

5. SHEAVES ON LOOP SPACES

In this section we consider the differential graded (dg) derived category of quasicoherent complexes on the derived loop space $\mathcal{L}X$ of a smooth Artin stack and the categorical action of S^1 induced by loop rotation. We begin in Section 5.1 with a review of the construction of dg derived categories of quasicoherent complexes on a derived stack. In Section 5.2, we describe categories of S^1 -equivariant sheaves on stacks in a fashion inspired by the Koszul duality picture of Goresky, Kottwitz and MacPherson [GKM]. After a review of the Koszul dual descriptions of de Rham modules and \mathcal{D} -modules on stacks in Section 5.3, we show in Section 5.4 that periodic S^1 -equivariant sheaves on the Hochschild space $\mathcal{H}X$ of a smooth Artin stack X are identified with periodic \mathcal{D} -modules on X .

5.1. Quasicoherent sheaves on a derived stack. In this section, we briefly summarize some of the key definitions and properties concerning dg categories and their construction from derived stacks. We refer the reader to [Ke, To2] for excellent overviews. We are indebted to Bertrand Toën for very helpful explanations. We refer to [L4, L5] for the theory of monoidal and symmetric monoidal quasicategories, specifically the notions of algebra objects, module categories over algebra objects, and tensor product of module objects in the commutative case (see also [SS] and references therein for the more familiar theory in the context of model categories).

A dg category over k is a category enriched over dg k -vector spaces. We remind the reader that throughout this paper k is assumed to be a field of characteristic zero; this significantly simplifies the associated homotopy theory. Our basic examples of dg categories are obtained by localizing quasi-isomorphisms in the category of complexes in an abelian category \mathcal{A} as explained by Keller [Ke] and Drinfeld [D]. We will consistently abuse standard terminology by referring to the result of this construction as the dg derived category of the underlying abelian category \mathcal{A} . It is worth pointing out from the start that not all of our dg derived categories will arise in this manner.

There is a notion of quasi-equivalence of dg categories, mimicking the notion of quasi-isomorphism of complexes: a quasiequivalence induces equivalences of homotopy categories. We would like to work with dg categories up to quasiequivalence. More formally, dg categories admit a model category structure [Ta] in which quasi-equivalences are the weak equivalences, giving rise to a quasicategory (the Dwyer-Kan simplicial localization) in which quasi-equivalences have been inverted. With this model structure in mind, we will construct many of our dg categories as limits of diagrams of dg categories.

To a derived stack Z , there is assigned a dg category $L_{\text{qcoh}}(Z)$ which we call the dg derived category of quasicoherent complexes on Z (see [To2, p.36]). Let us briefly recall the construction of $L_{\text{qcoh}}(Z)$.

First, consider a simplicial commutative k -algebra A , and the representable affine derived scheme $Z = \operatorname{Spec} A$. Via the normalization functor, we may think of A as a connective commutative differential graded algebra. With this understanding, we take $L_{\operatorname{qcoh}}(Z)$ to be the dg derived category of dg modules over A . In other words, we take the localization of dg modules over A with respect to quasi-isomorphisms.

In general, any derived stack Z can be written as a colimit of a diagram of affine derived schemes Z_\bullet . Then we take $L_{\operatorname{qcoh}}(Z)$ to be the limit of the corresponding diagram of dg categories $L_{\operatorname{qcoh}}(Z_\bullet)$. We can think of objects of $L_{\operatorname{qcoh}}(Z)$ as collections of quasicoherent complexes \mathcal{F}^\bullet on the terms Z_\bullet together with compatible collections of quasi-isomorphisms between their pullbacks under the diagram maps.

In our applications, we are interested in $L_{\operatorname{qcoh}}(Z)$ for derived Artin stacks Z with affine diagonal. In this case we can calculate $L_{\operatorname{qcoh}}(Z)$ by traditional simplicial descent of derived categories as explained for example in [BD, Section 7.4]. Choose an atlas

$$p : Z_0 \rightarrow Z,$$

such that Z_0 is an affine derived scheme, and consider the resolution of Z by the simplicial affine derived scheme Z_\bullet with k -simplices given by the fiber products

$$Z_k = Z_0 \times_Z^{\mathbf{L}} \cdots \times_Z^{\mathbf{L}} Z_0 \quad \text{with } k \text{ terms.}$$

Quasicoherent sheaves on the simplices form a cosimplicial dg category, and we take $L_{\operatorname{qcoh}}(Z)$ to be its totalization. Concretely, we can think of objects of $L_{\operatorname{qcoh}}(Z)$ as collections of quasicoherent sheaves \mathcal{F}^k on the simplices Z_k together with compatible quasi-isomorphisms between their pullbacks under the simplicial structure maps. In other words, sheaves on Z are described by modules for the cosimplicial commutative dg algebra of functions on the simplices.

Finally for a map $p : X \rightarrow Y$ we have the usual pullback functor $p^* : L_{\operatorname{qcoh}}(Y) \rightarrow L_{\operatorname{qcoh}}(X)$ and its right adjoint, the pushforward $p_* : L_{\operatorname{qcoh}}(X) \rightarrow L_{\operatorname{qcoh}}(Y)$.

We will be most interested in the dg derived category $L_{\operatorname{coh}}(Z)$ of quasi-coherent sheaves with finitely generated cohomology. Since one can define quasi-coherent sheaves with respect to smooth test maps, it makes sense to consider this property.

5.2. Models for equivariant sheaves. In this section, we consider equivariant sheaves on derived stacks with group actions. Our point of view is inspired by Koszul duality (specifically by [GKM, AP]). The basic idea is that to give a space Z with the action of a group G is the same as to give a space Z/G with a map to BG . The space Z/G is the total space of the Z -bundle over BG associated to the universal G -bundle EG and Z is the fiber of this bundle. After linearization, the constructions $Z \mapsto Z/G$ and $Z/G \mapsto Z$ become invariants over the linearization of G (which will be a group algebra) and coinvariants over the linearization of BG (which will be an equivariant cochain complex) respectively. In what follows, we will concentrate on the case $G = S^1$.

Goresky, Kottwitz and MacPherson [GKM] modify the grading conventions of BGG Koszul duality to obtain an equivalence between the homotopy theories of (bounded below) dg modules over the symmetric algebra

$$\mathbf{S} = H^*(BS^1) = k[u] \quad \text{with } |u| = 2$$

and (bounded below) dg modules over the exterior algebra

$$\mathbf{\Lambda} = H_{-*}(S^1) = k \oplus k \cdot \lambda \quad \text{with } |\lambda| = -1$$

preserving subcategories of complexes with finitely generated cohomology. Moreover, it is shown in [GKM] (in a geometric setting) that the \mathbf{S} -module of equivariant global sections of an S^1 -equivariant sheaf \mathcal{F} on an S^1 -space X corresponds to the \mathbf{A} -module of ordinary global sections of \mathcal{F} .

Now consider a derived stack Z with an action of S^1 . This consists of an action map

$$act : S^1 \times Z \rightarrow Z$$

with coherent associativity and unit axioms. Equivalently, we can give the derived stack Z/S^1 with a map to the classifying stack BS^1 , and an isomorphism

$$Z/S^1 \times_{BS^1} ES^1 \xrightarrow{\sim} Z$$

where $ES^1 \rightarrow BS^1$ is the universal S^1 -bundle. In cases of interest (such as loop spaces of Artin stacks) this action can be concretely modeled by giving a cyclic affine derived scheme. Intuitively, a sheaf on $Z \times S^1$ is simply a sheaf on Z with an automorphism, given by the monodromy along S^1 , so that an action of S^1 on Z manifests itself at the categorical level as an automorphism of the identity functor of $L_{\text{qcoh}}(Z)$.⁵

By definition, we take the dg derived category $L_{\text{qcoh}}(Z)^{S^1}$ of S^1 -equivariant quasicoherent sheaves on Z to be the dg derived category $L_{\text{qcoh}}(Z/S^1)$ of the quotient derived stack Z/S^1 . To obtain an explicit model, we present ES^1 by its standard simplicial model and calculate $Z/S^1 = Z \times_{S^1} ES^1$ by the Borel construction. We obtain $L_{\text{qcoh}}(Z/S^1)$ by taking the totalization of the corresponding cosimplicial dg category of quasicoherent sheaves on the simplices. Informally, the equivariant category is the homotopy equalizer of the identity and the monodromy operator on $L_{\text{qcoh}}(Z)$.

The equivariant derived category can be calculated by descent for the map $p : Z \rightarrow Z/S^1$. Namely, the canonical adjunction between p_* and p^* gives rise to a comonad on $L_{\text{qcoh}}(Z)$. Since p^* is conservative and we have enough colimits, the hypotheses of the Barr-Beck theorem are satisfied (see [L4] for the monadic formalism and Barr-Beck theorem in the context of quasicategories). The result is the following:

Proposition 5.1. *The category $L_{\text{qcoh}}(Z/S^1)$ is equivalent to the category of comodule objects in $L_{\text{qcoh}}(Z)$ for the coalgebra $p_* act^*$ in $\text{End}(L_{\text{qcoh}}(Z))$. Equivalently, it is realized as comodule objects for the coalgebra $(p \times act)_* \mathcal{O}_{S^1 \times Z}$ in $L_{\text{qcoh}}(Z \times Z)$ with its convolution monoidal structure.*

As a result, S^1 -equivariant quasicoherent sheaves on Z are the same thing as sheaves with a coaction of the cochain coalgebra $C^*(S^1)$ lifting its action on \mathcal{O}_Z .

As an illustration, consider the above descriptions of the equivariant category $L_{\text{qcoh}}(Z/S^1)$ in the case $Z = \text{Spec } k$ so that $Z/S^1 = BS^1$. It is traditional in this setting to regard comodule objects for the formal coalgebra $C^*(S^1)$ rather as module objects for the homology of the circle $\mathbf{A} = H_{-*}(S^1) = k \oplus k \cdot \lambda$ with $|\lambda| = -1$. Using the Dold-Kan correspondence, we can also identify $L_{\text{qcoh}}(BS^1)$ with the quasicategory of cyclic k -modules. As a result, we obtain a variant of the result of Dwyer-Kan [DK] giving an equivalence of quasicategories between \mathbf{A} -modules and cyclic k -modules.

More generally, let us spell out the statement of Proposition 5.1 in the case of an affine derived scheme $Z = \text{Spec } A$ with an S^1 -action. This will be the only form of the assertion used in what follows. First, we may consider A as a simplicial associative rather than commutative algebra since this does not affect the category of A -modules. Next, we may express the S^1 -action by considering A as a cyclic associative k -algebra since cyclic objects in a quasicategory

⁵More generally, a coaction $act^* : \mathcal{C} \rightarrow \mathcal{C} \otimes L_{\text{qcoh}}(S^1)$ on a dg category \mathcal{C} is equivalent to the data of an automorphism of the identity functor $m \in \text{Aut}(\text{Id}_{\mathcal{C}})$.

model objects with an S^1 -action. The S^1 -action on Z gives rise to a coaction of $C^*(S^1)$ on A , or equivalently an action of $\mathbf{\Lambda}$. We may therefore consider the algebra $A[\mathbf{\Lambda}]$ generated by A and $\mathbf{\Lambda}$ (in other words, obtained by adjoining the generator $\lambda \in \mathbf{\Lambda}^{-1}$ to A).

Corollary 5.2. *Let A be a cyclic commutative k -algebra, and let $A[\mathbf{\Lambda}]$ be the dg algebra generated by the S^1 -action on A . Then the equivariant dg derived category $L_{\text{qcoh}}(\text{Spec } A)^{S^1}$ is quasiequivalent to the dg derived category $A[\mathbf{\Lambda}] - \text{mod}$.*

5.3. Equivariant sheaves and de Rham modules. Let X be a smooth Artin stack with affine diagonal. Recall that its Hochschild space $\mathcal{H}X$ is a derived Artin stack and comes equipped with a canonical S^1 -action given by loop rotation. We are now ready to give a concrete description of the dg derived category $L_{\text{qcoh}}(\mathcal{H}X)^{S^1}$ of S^1 -equivariant quasicoherent complexes on $\mathcal{H}X$.

Choose a smooth simplicial presentation $X_\bullet \rightarrow X$ such that each of the simplices X_k is a smooth affine scheme. Since we assume that X has affine diagonal, we can choose a smooth affine cover $X_0 \rightarrow X$ with X_0 affine, and then take X_\bullet to be the Čech nerve with k -simplices given by the fiber products

$$X_k = X_0 \times_X \cdots \times_X X_0 \quad \text{with } k+1 \text{ factors.}$$

Consider the limit dg derived category $L_{\text{qcoh}}(\mathcal{H}X_\bullet)$ of compatible quasicoherent sheaves on the Hochschild spaces $\mathcal{H}X_k$ of each of the simplices X_k . Consider as well the limit dg derived category $L_{\text{coh}}(\mathcal{H}X_\bullet)$ of compatible coherent sheaves on the Hochschild spaces $\mathcal{H}X_k$ of each of the simplices X_k .

Proposition 5.3. *The canonical S^1 -equivariant map $\pi : \mathcal{H}X_\bullet \rightarrow \mathcal{H}X$ induces an S^1 -equivariant quasi-equivalence*

$$\pi^* : L_{\text{qcoh}}(\mathcal{H}X) \xrightarrow{\sim} L_{\text{qcoh}}(\mathcal{H}X_\bullet)$$

preserving subcategories of coherent objects.

Proof. By the descent description of the algebra $\mathcal{O}_{\mathcal{H}X}$ provided by Proposition 4.3 and the discussion thereafter, the dg derived category $L_{\text{qcoh}}(\mathcal{H}X)$ can be calculated as the limit dg derived category $L_{\text{qcoh}}(\mathcal{H}X_\bullet)$. \square

Let us continue to work with a smooth simplicial presentation $X_\bullet \rightarrow X$ such that each of the simplices X_k is a smooth affine scheme.

On each simplex X_k , consider the formal dg algebra of differential forms $\Omega_{X_k}^{-\bullet}$ placed in negative degrees, and the formal dg algebra $\Omega_{X_k}^{-\bullet}[d]$ obtained by adjoining the de Rham differential as an element of degree -1 . By an $\Omega_{X_k}^{-\bullet}$ -module (respectively, $\Omega_{X_k}^{-\bullet}[d]$ -module), we will mean a dg $\Omega_{X_k}^{-\bullet}$ -module (respectively, $\Omega_{X_k}^{-\bullet}[d]$ -module) that is quasicoherent as an \mathcal{O}_{X_k} -module.

We take the dg derived categories $L_{\text{qcoh}}(X, \Omega_X^{-\bullet})$ and $L_{\text{qcoh}}(X, \Omega_X^{-\bullet}[d])$ to be the limits of the corresponding cosimplicial dg derived categories. It is not difficult to check that the limit dg categories are independent of the choice of simplicial presentation $X_\bullet \rightarrow X$.

Theorem 5.4. *For a smooth Artin stack X , there are canonical quasiequivalences of dg derived categories*

$$\begin{aligned} L_{\text{qcoh}}(\mathcal{H}X) &\simeq \Omega_X^{-\bullet} - \text{mod} \\ L_{\text{qcoh}}(\mathcal{H}X)^{S^1} &\simeq \Omega_X^{-\bullet}[d] - \text{mod} \end{aligned}$$

preserving subcategories of coherent objects.

Proof. Observe that given a simplicial presentation $X_\bullet \rightarrow X$ such that each of the simplices X_k is a smooth affine scheme, all of the dg categories under consideration are calculated by taking the limit of the corresponding cosimplicial dg categories. Thus it suffices to prove the theorem for X a smooth affine scheme.

For X a smooth affine scheme, the first assertion is immediate from Proposition 4.3. This is simply a reformulation of the Hochschild-Kostant-Rosenberg theorem providing a quasi-isomorphism between Hochschild chains $C^{-\bullet}(\mathcal{O}_X)$ and the de Rham algebra $\Omega_X^{-\bullet}$.

For the second assertion, the S^1 -action on $\mathcal{H}X$ corresponds to a cyclic structure on $\mathcal{O}_{\mathcal{H}X}$. Under the Dold-Kan correspondence, the cyclic structure goes over to the Λ -structure on $C^{-\bullet}(\mathcal{O}_X)$ where the generator $\lambda \in \Lambda^{-1}$ acts by the Connes (homological) differential. Then under the Hochschild-Kostant-Rosenberg theorem, the Connes differential becomes identified with the de Rham differential on $\Omega_X^{-\bullet}$ (see [Lo]). Finally, Corollary 5.2 further identifies cyclic modules over $C^{-\bullet}(\mathcal{O}_X)$ with differential graded modules over the dg algebra $\Omega_X^{-\bullet}[d]$. This establishes the second assertion. \square

5.4. Koszul dual description. Let X be a smooth Artin stack with affine diagonal. In this section, we explain the close connection between S^1 -equivariant quasicoherent sheaves on the Hochschild space $\mathcal{H}X$ and filtered \mathcal{D} -modules on X . To make this precise, we will apply the Koszul duality functor of Goresky-Kottwitz-MacPherson [GKM]. This functor only differs from the Koszul duality functor of Kapranov [Ka1] and Beilinson-Drinfeld [BD] with respect to its grading convention.

Choose a smooth simplicial presentation $X_\bullet \rightarrow X$ such that each of the simplices X_k is a smooth affine scheme.

On each simplex X_k , consider the algebra of differential operators \mathcal{D}_{X_k} , and for each $i \geq 0$, the subsheaf $\mathcal{D}_{X_k, \leq i} \subset \mathcal{D}_{X_k}$ of differential operators of order at most i . Define the shifted Rees algebra \mathcal{R}_{X_k} to be the graded algebra

$$\mathcal{R}_{X_k} = \bigoplus_{i \geq 0} \mathcal{D}_{X_k, \leq i}[-2i]$$

where the graded piece $\mathcal{D}_{X_k, \leq i}$ is placed in cohomological degree $2i$.

Recall that $\Omega_X^{-\bullet}[d]$ denotes the de Rham algebra in negative degrees with the de Rham differential d adjoined as an element of degree -1 . If we think of \mathcal{R}_{X_k} as a left \mathcal{R}_{X_k} -module, then it admits a graded Koszul-de Rham complex \mathcal{K}_{X_k} . By definition, this is the dg \mathcal{O}_{X_k} -module

$$\mathcal{K}_{X_k} = (\Omega_{X_k}^{-\bullet}[d] \otimes_{\mathcal{O}_X} \mathcal{R}_{X_k}, \delta)$$

where the differential δ encodes the left action of \mathcal{R}_{X_k} on itself. We will think of \mathcal{K}_{X_k} as equipped with the obvious left action of $\Omega_{X_k}^{-\bullet}[d]$ and right action of \mathcal{R}_{X_k} .

On dg $\Omega_{X_k}^{-\bullet}[d]$ -modules M , and complexes of \mathcal{R}_{X_k} -modules N , we have adjoint functors

$$F_{X_k} : N \mapsto \mathrm{Hom}_{\mathcal{R}_{X_k}}(\mathcal{K}_{X_k}, N)[- \dim X_k / X] \quad G_{X_k} : M \mapsto M \otimes_{\Omega_{X_k}^{-\bullet}[d]} \mathcal{K}_{X_k}[\dim X_k / X]$$

Consider the dg derived categories $L_{\mathrm{coh}}(X_k, \Omega_{X_k}^{-\bullet}[d])$ and $L_{\mathrm{coh}}(X_k, \mathcal{R}_{X_k})$ of dg modules whose cohomology is finitely generated over the formal algebras $\Omega_{X_k}^{-\bullet}[d]$ and \mathcal{R}_{X_k} respectively. If we restrict to such modules, then the above functors descend to quasi-equivalences

$$F_{X_k} : L_{\mathrm{coh}}(X_k, \mathcal{R}_{X_k}) \xrightarrow{\sim} L_{\mathrm{coh}}(X_k, \Omega_{X_k}^{-\bullet}[d]) \quad G_{X_k} : L_{\mathrm{coh}}(X_k, \Omega_{X_k}^{-\bullet}[d]) \xrightarrow{\sim} L_{\mathrm{coh}}(X_k, \mathcal{R}_{X_k})$$

For more details, the reader could consult Kapranov [Ka1].

Finally, define the dg derived categories $L_{\mathrm{coh}}(X, \Omega_X^{-\bullet}[d])$ and $L_{\mathrm{coh}}(X, \mathcal{R}_X)$ to be the limits of the corresponding cosimplicial dg derived categories. Applying the above functors on each

of the simplices leads to analogous functors F_X and G_X on the limit dg categories. For details of the following assertion, the reader could consult [BD, Section 7.5].

Theorem 5.5. *The functors F_X and G_X provide quasi-equivalences of dg derived categories*

$$F_X : L_{\text{coh}}(X, \mathcal{R}_X) \xrightarrow{\sim} L_{\text{coh}}(X, \Omega_X^{-\bullet}[d]) \quad G_X : L_{\text{coh}}(X, \Omega_X^{-\bullet}[d]) \xrightarrow{\sim} L_{\text{coh}}(X, \mathcal{R}_X)$$

Putting together Theorems 5.4 and 5.5, we immediately obtain the following.

Corollary 5.6. *There is a canonical quasi-equivalence of dg derived categories*

$$L_{\text{coh}}(\mathcal{H}X)^{S^1} \simeq L_{\text{coh}}(X, \mathcal{R}_X).$$

5.5. Periodic sheaves and \mathcal{D} -modules. Finally, we will consider the periodic, or localized, version of the above picture. The category $L_{\text{coh}}(\mathcal{H}X/S^1)$ (much as every equivariant category) is linear over the equivariant cohomology ring of a point

$$\mathbf{S} = H^*(BS^1) = k[u] \quad \text{with } |u| = 2,$$

the Koszul dual of the homology ring \mathbf{A} of S^1 .

For a \mathbf{S} -linear dg category \mathcal{C} we may consider the corresponding periodic category, in which we localize \mathcal{C} with respect to the action of the central element u (following [To1]):

$$\mathcal{C}_{\text{per}}^{S^1} = \mathcal{C}^{S^1} \otimes_{k[u]} k[u, u^{-1}].$$

Since u has cohomological degree 2, the morphism complexes in the new category are 2-periodic.

One may also periodicize a dg category by force: for a dg category \mathcal{C} linear over k , we can always extend scalars to obtain a new category

$$\mathcal{C} \otimes_k k[u, u^{-1}],$$

where again the morphism complexes are 2-periodic. (More precisely, we take the pre-triangulated envelope of the naive tensor product.)

Corollary 5.7. *There is a canonical quasi-equivalence of periodic dg categories*

$$L_{\text{coh}}(\mathcal{H}X)_{\text{per}}^{S^1} \simeq L_{\text{coh}}(X, \mathcal{D}_X) \otimes_k k[u, u^{-1}]$$

Proof. Consider the central element $t \in \mathcal{R}_X^2$ given by the unit of $\mathcal{O}_X \subset \mathcal{D}_{X, \leq 1}$ under the identification

$$\mathcal{D}_{X, \leq 1}[-2] = \mathcal{R}_X^2.$$

We consider \mathcal{R}_X as an \mathbf{S} -algebra, where $u \in \mathbf{S}$ acts by multiplication by t . Inverting t in the graded algebra \mathcal{R}_X we obtain the periodic version of the algebra \mathcal{D}_X :

$$\mathcal{R}_X[t^{-1}] \simeq \mathcal{D}_X \otimes_k k[t, t^{-1}].$$

Inverting t on the level of module categories, we obtain an equivalence

$$L_{\text{coh}}(X, \mathcal{R}_X) \otimes_{k[t]} k[t, t^{-1}] \simeq L_{\text{coh}}(X, \mathcal{D}_X) \otimes_k k[t, t^{-1}].$$

(The equivalence follows from the existence of good filtrations on coherent \mathcal{D} -modules.)

In order to conclude, we need to identify the action of $u \in H^2(BS^1)$ on the S^1 -equivariant category $L_{\text{qcoh}}(\mathcal{H}X)^{S^1}$ with the action of the central element $t \in \mathcal{R}_X^2$ on Rees modules under the quasi-equivalence of Corollary 5.6. This identification is a consequence of the evident compatibility between the Koszul duality between \mathbf{A} and \mathbf{S} and the Koszul duality between $\Omega_X^{-\bullet}[d] = \mathcal{O}_{\mathcal{H}X}[\mathbf{A}]$ and the \mathbf{S} -algebra $\mathbf{S} \hookrightarrow \mathcal{R}_X$. □

Remark 5.8. To obtain a \mathbf{Z} -graded version of the above corollary, we should work in a “mixed” setting with an extra grading direction.

6. STEINBERG VARIETIES AS LOOP SPACES

In this section, we apply the preceding results to a concrete example in representation theory. We show how to relate two prominently appearing categories associated to a complex reductive group G . The first is the category of equivariant coherent sheaves on the Steinberg variety of G . The second is the category of Harish-Chandra bimodules of G with strictly trivial infinitesimal character. Via Beilinson-Bernstein localization, we can identify the latter with equivariant \mathcal{D} -modules on the flag variety of G . We refer the interested reader to the overview in Section 2 for a brief discussion of the importance of these categories in representation theory, and in particular for their interpretation as Langlands parameters for representations of the Langlands dual group. Our aim here is to see how to recover the second category from the first via the geometry of loop spaces and S^1 -equivariant localization.

One of the complications in explaining this picture is the fact that there is not one result but rather a family of results parametrized by a product of universal Cartan groups $H \times H$. (These parameters correspond to the infinitesimal character of representations of the Langlands dual group.) To deal with this, we introduce more and more local notions of what one might mean by the Steinberg variety. First and foremost, there is the global Steinberg variety $\mathcal{S}t$ whose quotient $\mathcal{S}t/G$ admits a realization as a loop space. Then for each (interesting) value of the parameter $(\alpha, w\alpha)$, we have the local Steinberg variety $\mathcal{S}t_{\alpha, w\alpha}$ obtained by completing the global Steinberg variety $\mathcal{S}t$ with respect to a neighborhood of the parameter $(\alpha, w\alpha)$. Finally, we have the formal Steinberg variety $\widehat{\mathcal{S}t}_{\alpha, \beta}$ obtained by completing the local Steinberg variety $\mathcal{S}t_{\alpha, w\alpha}$ with respect to its unipotent directions.

It is the formal Steinberg space $\widehat{\mathcal{S}t}_{\alpha, \beta}/G$ that turns out to be a Hochschild space, while the local Steinberg space $\mathcal{S}t_{\alpha, \beta}/G$ is what appears most commonly in representation theory. The general results of previous sections go so far as to relate equivariant \mathcal{D} -modules on certain flag varieties to coherent sheaves on the equivariant formal Steinberg space $\widehat{\mathcal{S}t}_{\alpha, \beta}/G$. To complete the bridge to representation theory, we check that the restriction of coherent sheaves from the local space $\mathcal{S}t_{\alpha, \beta}/G$ to the formal space $\widehat{\mathcal{S}t}_{\alpha, \beta}/G$ is an equivalence.

In the first two sections that follow, we explain the above story for the trivial parameter. In the third and final section, we explain the natural generalization for arbitrary parameters.

We first set some notation. Let G be a connected reductive complex algebraic group with Lie algebra \mathfrak{g} , and let \mathcal{B} be the flag variety of G parameterizing Borel subgroups $B \subset G$. For each $B \in \mathcal{B}$, we have the Cartan quotient $H = B/U$ where $U \subset B$ is the unipotent radical. The natural G -action on \mathcal{B} by conjugation canonically identifies the Cartan quotients for different B , and so we call H the universal Cartan of G .

6.1. The Steinberg space. All of the spaces we will consider naturally live over the classifying space pt/G . (To avoid confusion with our notation for Borel subgroups, we will denote the classifying space by pt/G instead of the customary BG .) What follows is a brief introduction to our *dramatis personae*.

6.1.1. The adjoint group. Let G/G be the adjoint group defined by taking the quotient of G by the adjoint action of G on itself. As we have seen, as a group space over pt/G , it is isomorphic to the loop space $\mathcal{L}(pt/G)$.

6.1.2. The Grothendieck-Springer space. Let \widetilde{G} be the the Grothendieck-Springer variety of pairs of an element $g \in G$ and a Borel subgroup $B \in \mathcal{B}$ such that $g \in B$. We refer to the quotient \widetilde{G}/G by the adjoint action of G as the Grothendieck-Springer space or equivariant Grothendieck-Springer variety.

Since G acts transitively on \mathcal{B} and the stabilizer of $B \in \mathcal{B}$ is precisely B , we see that \tilde{G}/G is canonically isomorphic to the adjoint group B/B , for any $B \in \mathcal{B}$. Thus it is isomorphic to the loop space $\mathcal{L}(pt/B)$ as a group space over pt/B .

Let $p : \tilde{G}/G \rightarrow G/G$ be the projection $p(g, B) = g$. If we think of \tilde{G}/G as the loop space $\mathcal{L}(pt/B)$, and G/G as the loop space $\mathcal{L}(pt/G)$, then p is the map on loops induced by the natural fibration $pt/B \rightarrow pt/G$.

Let $\pi : \tilde{G}/G \rightarrow \mathcal{B}/G$ be the other projection $\pi(g, B) = B$. Again, if we think of \tilde{G}/G as the loop space $\mathcal{L}(pt/B)$, and \mathcal{B}/G as the classifying space pt/B , then π is the natural projection $\mathcal{L}(pt/B) \rightarrow pt/B$ obtained by evaluating a loop at $1 \in S^1$.

6.1.3. The flag space. Consider the diagonal G -action on the product $\mathcal{B} \times \mathcal{B}$ of two copies of the flag variety. We refer to the corresponding quotient stack $(\mathcal{B} \times \mathcal{B})/G$ as the flag space or equivariant flag variety. For a fixed element $(B_1, B_2) \in \mathcal{B} \times \mathcal{B}$, we have a canonical identification

$$B_1 \backslash G / B_2 \xrightarrow{\sim} (\mathcal{B} \times \mathcal{B})/G \quad g \mapsto (B_1, gB_2g^{-1}).$$

6.1.4. The global Steinberg space. By the global Steinberg variety, we mean the fiber product

$$St = \tilde{G} \times_G \tilde{G}.$$

By a simple dimension count, the above ordinary fiber product coincides with its derived enhancement. We refer to the corresponding quotient stack

$$St/G = (\tilde{G} \times_G \tilde{G})/G \simeq (\tilde{G}/G \times_{G/G} \tilde{G}/G)$$

as the global Steinberg space or equivariant global Steinberg variety.

The natural projection $\pi : St/G \rightarrow (\mathcal{B} \times \mathcal{B})/G$ realizes St/G as a group-space over $(\mathcal{B} \times \mathcal{B})/G$. Concretely, we have two universal Borel subgroups

$$B_1^{univ}, B_2^{univ} \subset (\mathcal{B} \times \mathcal{B} \times G)/G,$$

and St/G is their intersection

$$St/G = B_1^{univ} \cap B_2^{univ}.$$

Again thanks to a dimension count, the above ordinary intersection coincides with its derived enhancement.

For our purposes, the fundamental viewpoint on the global Steinberg space is given by its natural realization as the loop space of the flag space.

Theorem 6.1. *We have a canonical identification*

$$St/G \simeq \mathcal{L}((\mathcal{B} \times \mathcal{B})/G) \simeq \mathcal{L}(B \backslash G / B)$$

of group spaces over $(\mathcal{B} \times \mathcal{B})/G \simeq B \backslash G / B$.

Proof. Recall that by definition, the inertia stack $I((\mathcal{B} \times \mathcal{B})/G)$ is the underived mapping stack $\text{Hom}(S^1, (\mathcal{B} \times \mathcal{B})/G)$. It is immediate from the definitions that $I((\mathcal{B} \times \mathcal{B})/G)$ is precisely the global Steinberg space St/G . Thus to establish the theorem, we must see that the loop space $\mathcal{L}((\mathcal{B} \times \mathcal{B})/G)$ coincides with $I((\mathcal{B} \times \mathcal{B})/G)$. In other words, we must see that the derived structure of $\mathcal{L}((\mathcal{B} \times \mathcal{B})/G)$ is trivial. But this is immediate from Proposition 4.2. \square

6.1.5. *The Steinberg space.* Let $p : \mathcal{St} \rightarrow H \times H$ be the natural projection to the product of universal Cartans. We call $H \times H$ the monodromic base, and refer to p as the monodromic projection.

Let $\widehat{H}_e \times \widehat{H}_e$ be the formal neighborhood of the unit $(e, e) \in H \times H$, and define the Steinberg variety $\mathcal{St}_{e,e}$ to be the inverse image

$$\mathcal{St}_{e,e} = p^{-1}(\widehat{H}_e \times \widehat{H}_e).$$

We refer to the corresponding quotient stack $\mathcal{St}_{e,e}/G$ as the Steinberg space or equivariant Steinberg variety.

Since p is a group homomorphism, $\mathcal{St}_{e,e}/G$ is a group space over $(\mathcal{B} \times \mathcal{B})/G$.

Remark 6.2. It is worth pointing out that each component of the monodromic projection p more naturally takes values in the adjoint group H/H . Of course, the conjugation action here is trivial, thus we have $H/H \simeq H \times pt/H$, and so we were able to project to the factor H . It is likely that the extra parameters coming from the factor pt/H will contribute meaningful deformations of the current story.

6.1.6. *The formal Steinberg space.* We define the formal Steinberg variety $\widehat{\mathcal{St}}_{e,e}$ to be the formal completion of the Steinberg variety $\mathcal{St}_{e,e}$ along the trivial section $(\mathcal{B} \times \mathcal{B}) \rightarrow \mathcal{St}_{e,e}$, or equivalently, the formal completion of the global Steinberg variety \mathcal{St} along the trivial section $(\mathcal{B} \times \mathcal{B}) \rightarrow \mathcal{St}$. We refer to the corresponding quotient stack $\widehat{\mathcal{St}}_{e,e}/G$ as the formal Steinberg space or equivariant formal Steinberg variety.

We have a diagram of group-spaces

$$\widehat{\mathcal{St}}_{e,e}/G \rightarrow \mathcal{St}_{e,e}/G \rightarrow \mathcal{St}/G$$

over $(\mathcal{B} \times \mathcal{B})/G$. The first is the formal group, the second is the relative completion in the directions of the monodromic base, and the last is the global group.

By construction, Theorem 6.1 immediately implies the following.

Corollary 6.3. *We have a canonical identification*

$$\widehat{\mathcal{St}}_{e,e}/G \simeq \mathcal{H}((\mathcal{B} \times \mathcal{B})/G) \simeq \mathcal{H}(B \backslash G/B)$$

of formal groups over $(\mathcal{B} \times \mathcal{B})/G \simeq B \backslash G/B$.

6.2. **Harish-Chandra bimodules.** Let $\mathfrak{U}(\mathfrak{g} \times \mathfrak{g})$ denote the universal enveloping algebra of the Lie algebra $\mathfrak{g} \times \mathfrak{g}$. Consider the Harish-Chandra pair $(\mathfrak{U}(\mathfrak{g} \times \mathfrak{g}), G)$, and the corresponding category of Harish-Chandra bimodules of G with trivial infinitesimal character. By definition, these are modules over $(\mathfrak{U}(\mathfrak{g} \times \mathfrak{g}), G)$ which are finitely generated over $\mathfrak{U}(\mathfrak{g} \times \mathfrak{g})$ and on which the center $\mathfrak{Z}(\mathfrak{g} \times \mathfrak{g}) \subset \mathfrak{U}(\mathfrak{g} \times \mathfrak{g})$ acts via the trivial character.

By Beilinson-Bernstein localization, the abelian category of Harish Chandra bimodules of G with (strictly) trivial infinitesimal character is equivalent to the abelian category of G -equivariant \mathcal{D} -modules on $\mathcal{B} \times \mathcal{B}$. One should beware that the naive corresponding dg derived category of Harish-Chandra bimodules is *not* equivalent to the natural G -equivariant dg derived category of \mathcal{D} -modules on $\mathcal{B} \times \mathcal{B}$. It is the latter category that plays a more prominent role in representation theory (for example, by [BGS] it is Koszul dual to the derived category of representations with generalized trivial infinitesimal character, and by the work of [So] it arises as parameters for representations of the dual group), and thus we will proceed with it in mind as our model for the dg derived category of Harish-Chandra bimodules. Alternatively, one could take the appearance of Harish-Chandra bimodules as purely motivational, and understand that

it is the G -equivariant dg derived category of \mathcal{D} -modules on $\mathcal{B} \times \mathcal{B}$ that we are most interested in.

We have seen in Corollary 6.3 that the formal Steinberg space $\widehat{\mathcal{S}t}_{e,e}/G$ is the Hochschild space of the flag space $(\mathcal{B} \times \mathcal{B})/G$. Thus applying our general results, we have a canonical quasi-equivalence of dg derived categories

$$L_{\text{coh}}(\widehat{\mathcal{S}t}_{e,e}/G)^{S^1} \xrightarrow{\sim} L_{\text{coh}}(\mathcal{R}_{(\mathcal{B} \times \mathcal{B})/G}).$$

Since the formal Steinberg space $\widehat{\mathcal{S}t}_{e,e}/G$ is not a commonly appearing object in representation theory, the above identification is not completely satisfactory. But thanks to the following lemma, we can go one step further and relate these categories to coherent sheaves on the Steinberg space $\mathcal{S}t_{e,e}/G$ itself. Note that the following lemma is the one place where we need to work with coherent sheaves rather than quasicoherent sheaves.

Lemma 6.4. *The natural restriction map*

$$L_{\text{coh}}(\mathcal{S}t_{e,e}/G) \rightarrow L_{\text{coh}}(\widehat{\mathcal{S}t}_{e,e}/G)$$

is an S^1 -equivariant equivalence.

Proof. The map $\widehat{\mathcal{S}t}_{e,e}/G \rightarrow \mathcal{S}t_{e,e}/G$ is clearly S^1 -equivariant, and so the S^1 -equivariance of the lemma is immediate.

To see the restriction is an equivalence, fix a Borel subgroup $B \in \mathcal{B}$ with unipotent radical $U \subset B$, and consider the formal completion \widehat{B}_U along U and the formal group \widehat{B} . By base change, it suffices to show that the restriction functor

$$L_{\text{coh}}(\widehat{B}_U/B) \rightarrow L_{\text{coh}}(\widehat{B}/B).$$

is an equivalence. Observe that each of the above categories can be thought of as a category of representations of B equipped with extra structure.

Fix a generic one-parameter subgroup $\mathbb{G}_m \subset B$ so that the induced conjugation action of \mathbb{G}_m on the unipotent radical U contracts it to the identity $e \in U$. It is straightforward to check that the above restriction functor is equal to the completion functor with respect to the weights of the \mathbb{G}_m -action. Conversely, we can define an inverse functor

$$L_{\text{coh}}(\widehat{B}/B) \rightarrow L_{\text{coh}}(\widehat{B}_U/B)$$

by taking the \mathbb{G}_m -finite vectors of any object. \square

Putting together Corollary 6.3, Lemma 6.4 and our general results, we arrive at our goal as summarized in the following statements.

Theorem 6.5. *There is a canonical quasi-equivalence of dg derived categories*

$$L_{\text{coh}}(\mathcal{S}t_{e,e}/G)^{S^1} \xrightarrow{\sim} L_{\text{coh}}(\mathcal{R}_{(\mathcal{B} \times \mathcal{B})/G}).$$

Corollary 6.6. *There is a canonical quasi-equivalence of dg derived categories*

$$L_{\text{coh}}(\mathcal{S}t_{e,e}/G)_{\text{per}}^{S^1} \xrightarrow{\sim} L_{\text{coh}}(\mathcal{D}_{(\mathcal{B} \times \mathcal{B})/G}) \otimes_{\mathbb{C}} \mathbb{C}[u, u^{-1}].$$

6.3. General monodromicities. In this section, we establish results for a general monodromic parameter analogous to those of the previous sections.

Recall that we have the monodromic projection $p : \mathcal{St} \rightarrow H \times H$, and that the Steinberg variety $\mathcal{St}_{e,e}$ is defined to be the inverse image under p of the formal group $\widehat{H}_e \times \widehat{H}_e$. Given an arbitrary pair $(\alpha, \beta) \in H \times H$, we denote its formal neighborhood by $\widehat{H}_\alpha \times \widehat{H}_\beta$, and define the monodromic Steinberg variety $\mathcal{St}_{\alpha,\beta}$ to be the inverse image

$$\mathcal{St}_{\alpha,\beta} = p^{-1}(\widehat{H}_\alpha \times \widehat{H}_\beta).$$

We refer to the corresponding quotient stack $\mathcal{St}_{\alpha,\beta}/G$ as the monodromic Steinberg space or equivariant monodromic Steinberg variety.

Our aim here is to give a loop space interpretation of coherent sheaves on $\mathcal{St}_{\alpha,\beta}/G$.

Let W be the Weyl group of G .

Lemma 6.7. *The monodromic Steinberg space $\mathcal{St}_{\alpha,\beta}/G$ is nonempty if and only if there is $w \in W$ such that $\beta = w\alpha$.*

Proof. For $(g, B_1, B_2) \in \mathcal{St}$, let α be the class of g in B_1/U_1 and β the class of g in B_2/U_2 . We may conjugate B_1 to B_2 to see that α must be conjugate to β . \square

Now fix $\alpha \in H$, and consider the monodromic Steinberg variety $\mathcal{St}_{\alpha,w\alpha}$, for some $w \in W$.

Let $\mathcal{O}_\alpha \subset G$ denote the semisimple conjugacy class corresponding to α . Fix once and for all an element $\tilde{\alpha} \in \mathcal{O}_\alpha$, and let $G(\alpha) \subset G$ denote its centralizer. In general, $G(\alpha)$ is reductive, and often turns out to be a Levi subgroup.

We will affix the symbol (α) to our usual notation when referring to objects associated to $G(\alpha)$. So for example, we write $\mathcal{B}(\alpha)$ for the flag variety of $G(\alpha)$, and $\mathcal{St}(\alpha)$ for its global Steinberg variety. Furthermore, we have the monodromic projection $p(\alpha) : \mathcal{St}(\alpha) \rightarrow H \times H$, the monodromic Steinberg variety $\mathcal{St}(\alpha)_{\alpha,\alpha}$ given by the inverse image

$$\mathcal{St}(\alpha)_{\alpha,\alpha} = p(\alpha)^{-1}(\widehat{H}_\alpha \times \widehat{H}_\alpha),$$

and the corresponding monodromic Steinberg space $\mathcal{St}(\alpha)_{\alpha,\alpha}/G(\alpha)$.

Theorem 6.8. *For each $w \in W$, we have a canonical S^1 -equivariant identification of monodromic Steinberg spaces*

$$\mathcal{St}_{\alpha,w\alpha}/G \simeq \mathcal{St}(\alpha)_{\alpha,\alpha}/G(\alpha).$$

Proof. For each $w \in W$, one can check that there is a map

$$\begin{aligned} \mathcal{St}(\alpha)_{\alpha,\alpha} &\rightarrow \mathcal{St}_{\alpha,w\alpha} \\ (g, B(\alpha)_1, B(\alpha)_2) &\mapsto (g, B_1, B_2) \end{aligned}$$

uniquely characterized by the properties:

$$\begin{aligned} B_1 \cap G(\alpha) &= B(\alpha)_1 & B_2 \cap G(\alpha) &= B(\alpha)_2 \\ [g]_1 &\in \widehat{H}_\alpha & [g]_2 &\in \widehat{H}_{w\alpha} \end{aligned}$$

where $[g]_1, [g]_2$ denote the classes of g in $B_1/U_1, B_2/U_2$ respectively.

Passing to the respective quotients gives the sought-after isomorphism. \square

By Theorem 6.8, to understand S^1 -equivariant coherent sheaves on $\mathcal{St}_{\alpha,w\alpha}/G$ it suffices to understand them on $\mathcal{St}(\alpha)_{\alpha,\alpha}/G(\alpha)$. This is very close to a problem we have already solved. Namely, consider the Steinberg space $\mathcal{St}(\alpha)_{e,e}/G(\alpha)$ still for the group $G(\alpha)$, but now for the

trivial monodromic parameter. Applying Theorem 6.5, we obtain a canonical quasi-equivalence of dg derived categories

$$L_{\text{coh}}(\mathcal{S}t(\alpha)_{e,e}/G(\alpha))^{S^1} \xrightarrow{\sim} L_{\text{coh}}(\mathcal{R}_{(\mathcal{B}(\alpha) \times \mathcal{B}(\alpha))/G(\alpha)}).$$

Moreover, multiplication by the fixed central element $\tilde{\alpha} \in G(\alpha)$ provides an isomorphism

$$\begin{aligned} \mathcal{S}t(\alpha)_{e,e}/G(\alpha) &\xrightarrow{\sim} \mathcal{S}t(\alpha)_{\alpha,\alpha}/G(\alpha) \\ (g, B(\alpha)_1, B(\alpha)_2) &\mapsto (\tilde{\alpha}g, B(\alpha)_1, B(\alpha)_2). \end{aligned}$$

Thus putting the two statements together, we should be able to conclude something about S^1 -equivariant coherent sheaves on $\mathcal{S}t(\alpha)_{\alpha,\alpha}/G(\alpha)$.

Unfortunately, the above isomorphism is *not* S^1 -equivariant, and in general S^1 -equivariant coherent sheaves on the two sides will *not* be the same. What is true is that it induces a quasi-equivalence on the level of *localized* S^1 -equivariant categories. Applying Corollary 6.6, we have a canonical quasi-equivalence of dg derived categories

$$L_{\text{coh}}(\mathcal{S}t(\alpha)_{e,e}/G(\alpha))_{\text{per}}^{S^1} \xrightarrow{\sim} L_{\text{coh}}(\mathcal{D}_{(\mathcal{B}(\alpha) \times \mathcal{B}(\alpha))/G(\alpha)}) \otimes_{\mathbf{C}} \mathbf{C}[u, u^{-1}].$$

Thanks to the rigidity of \mathcal{D} -modules (as opposed to Rees modules), we have the following.

Theorem 6.9. *There is a canonical quasi-equivalence of dg derived categories*

$$L_{\text{coh}}(\mathcal{S}t(\alpha)_{e,e}/G(\alpha))_{\text{per}}^{S^1} \xrightarrow{\sim} L_{\text{coh}}(\mathcal{S}t(\alpha)_{\alpha,\alpha}/G(\alpha))_{\text{per}}^{S^1}$$

Proof. Consider the isomorphism

$$\mathcal{S}t(\alpha)_{e,e}/G(\alpha) \xrightarrow{\sim} \mathcal{S}t(\alpha)_{\alpha,\alpha}/G(\alpha)$$

given by multiplication by the fixed central element $\tilde{\alpha} \in G(\alpha)$. Let us compare the universal monodromies of the respective S^1 -actions under the induced quasi-equivalence

$$L_{\text{coh}}(\mathcal{S}t(\alpha)_{e,e}/G(\alpha)) \xrightarrow{\sim} L_{\text{coh}}(\mathcal{S}t(\alpha)_{\alpha,\alpha}/G(\alpha)).$$

By construction, we have an identity

$$m_{\alpha} = m_e \circ \varphi_{\tilde{\alpha}}$$

where m_{α} is the universal monodromy of the right hand side, m_e is that of the left hand side, and $\varphi_{\tilde{\alpha}}$ is the automorphism of the identity functor given by the central element $\tilde{\alpha} \in G(\alpha)$.

Since automorphisms of the identity functor mutually commute, the automorphism $\varphi_{\tilde{\alpha}}$ passes to the S^1 -equivariant category with respect to the monodromy m_e and further to its localization. We claim that after taking the localized S^1 -equivariant category with respect to the monodromy m_e , the automorphism $\varphi_{\tilde{\alpha}}$ acts *trivially*. If so, then every localized S^1 -equivariant object or morphism with respect to m_e is canonically a localized S^1 -equivariant object or morphism with respect to m_{α} and vice versa. Thus to prove the theorem, it suffices to establish the above claim.

To prove the claim, let us consider the automorphism $\varphi_{\tilde{\alpha}}$ under the identification

$$L_{\text{coh}}(\mathcal{S}t(\alpha)_{e,e}/G(\alpha))_{\text{per}}^{S^1} \xrightarrow{\sim} L_{\text{coh}}(\mathcal{D}_{(\mathcal{B}(\alpha) \times \mathcal{B}(\alpha))/G(\alpha)}) \otimes_{\mathbf{C}} \mathbf{C}[u, u^{-1}].$$

Given any object of the stack $(\mathcal{B}(\alpha) \times \mathcal{B}(\alpha))/G(\alpha)$, its automorphism group contains a maximal torus $T(\alpha) \subset G(\alpha)$. Thus the central element $\tilde{\alpha} \in G(\alpha)$ can be connected to the identity by a path in the automorphism group. Thus it acts trivially on any \mathcal{D} -module on $(\mathcal{B}(\alpha) \times \mathcal{B}(\alpha))/G(\alpha)$, and the claim follows. \square

Now returning to our original problem, Theorems 6.8 and 6.9 and Corollary 6.6 immediately imply the following.

Corollary 6.10. *For each $w \in W$, there is a canonical quasi-equivalence of dg derived categories*

$$L_{\text{coh}}(\mathcal{S}t_{\alpha, w\alpha}/G)_{\text{per}}^{S^1} \xrightarrow{\sim} L_{\text{coh}}(\mathcal{D}_{(\mathcal{B}(\alpha) \times \mathcal{B}(\alpha))/G(\alpha)}) \otimes_{\mathbf{C}} \mathbf{C}[u, u^{-1}].$$

7. LANGLANDS PARAMETER SPACES

This section contains our main intended application of the relation between localized S^1 -equivariant coherent sheaves on Hochschild spaces and \mathcal{D} -modules. Given an involution η of the complex reductive group G , we introduce a stack $\mathcal{S}t^\eta/G$ which we call the global Langlands parameter space. Our aim in this section is to explain the relationship between localized S^1 -equivariant coherent sheaves on $\mathcal{S}t^\eta/G$ and \mathcal{D} -modules on certain flag spaces. For further motivation, we refer the interested reader to the overview and applications described in Sections 2 and 3. Roughly speaking, the global Langlands space $\mathcal{S}t^\eta/G$ plays an analogous role to that of the global Steinberg space $\mathcal{S}t/G$ but now for *real forms* of the dual group G^\vee . The flag spaces that arise in this section are precisely the geometric parameter spaces of Adams, Barbasch and Vogan [ABV], which appear in the geometry of Vogan duality and Soergel's conjecture.

Many of the constructions and arguments of this section are direct generalizations of those of Section 6. In fact, the results of Section 6 are the special case when we take G to be a product $G_o \times G_o$ and η to be the involution that switches the factors. We have chosen to separate this case and explain it previously on its own for two reasons. First, this case corresponds to the representation theory of the complex dual group G_o^\vee itself considered as a real group, and so being of special significance warrants its own statements. Second, the proofs in the general case are more complicated notationally but not conceptually.

In parallel with Section 6, the results of this section will be parametrized by a (single) universal Cartan group H . To deal with this, we introduce more and more local notions of the global Langlands parameter space $\mathcal{S}t^\eta/G$. For each value of the parameter α , we have the local Langlands parameter space $\mathcal{S}t_\alpha^\eta/G$ obtained by completing the global Langlands parameter space $\mathcal{S}t^\eta/G$ with respect to a neighborhood of the parameter α . Furthermore, we have the formal Langlands parameter space $\widehat{\mathcal{S}t}_\alpha^\eta/G$ obtained by completing the local Langlands parameter space $\mathcal{S}t_\alpha^\eta/G$ with respect to its unipotent directions.

In continued parallel with Section 6, it is the formal Langlands parameter space $\widehat{\mathcal{S}t}_\alpha^\eta/G$ that turns out to be a Hochschild space, while the local Langlands parameter space $\mathcal{S}t_\alpha^\eta/G$ is what plays a significant role in representation theory as discussed in Section 3. Our general results go so far as to relate equivariant \mathcal{D} -modules on certain flag varieties to coherent sheaves on the formal Langlands parameter space $\widehat{\mathcal{S}t}_\alpha^\eta/G$. To complete the bridge to our desired applications, we check that the restriction of coherent sheaves from the local space $\mathcal{S}t_\alpha^\eta/G$ to the formal space $\widehat{\mathcal{S}t}_\alpha^\eta/G$ is an equivalence.

Because of the close parallels with Section 6, we have kept the arguments of this section brief; they often simply refer to the analogous arguments of the preceding section. In the first two sections that follow, we explain the story for the trivial parameter. In the third and final section, we explain the natural generalization for arbitrary parameters.

We continue with the notation of the previous section so for example, G is a complex reductive group with flag variety \mathcal{B} . Fix once and for all an algebraic involution η of G . Form the semidirect product or L -group

$$G_\eta = G \rtimes \mathbf{Z}/2\mathbf{Z}$$

where the nontrivial element of $\mathbf{Z}/2\mathbf{Z}$ acts on G by η . We identify G with the identity component of G_η , and write $G_\eta \setminus G$ for the other component.

7.1. The Langlands space. Many of the spaces we will consider naturally live over the classifying space pt/G_η . It sometimes will be useful to fix the trivial $\mathbf{Z}/2\mathbf{Z}$ -bundle on a point, and consider the resulting base change diagram

$$\begin{array}{ccc} pt/G & \rightarrow & pt/G_\eta \\ \downarrow & & \downarrow \\ pt & \rightarrow & pt/(\mathbf{Z}/2\mathbf{Z}) \end{array}$$

In general, for a space Y living over pt/G_η , we will refer to the base change

$$Y_{\text{rigid}} = Y \times_{pt/G_\eta} pt/G$$

as the $\mathbf{Z}/2\mathbf{Z}$ -rigidification of Y .

As in the preceding section, we first introduce the *dramatis personae* of our story.

7.1.1. The adjoint group. The loop space $\mathcal{L}(pt/G_\eta)$ is nothing more than the adjoint group G_η/G_η . Observe that $\mathcal{L}(pt/G_\eta)$ has two connected components

$$\mathcal{L}(pt/G_\eta) = G/G_\eta \cup (G_\eta \setminus G)/G_\eta$$

corresponding to $\mathbf{Z}/2\mathbf{Z} = \pi_0(G_\eta) \simeq \pi_1(pt/G_\eta)$. The $\mathbf{Z}/2\mathbf{Z}$ -rigidification of $\mathcal{L}(pt/G_\eta)$ is simply the quotient G_η/G by the restriction of the conjugation action.

7.1.2. The symmetric flag space. Consider the diagonal G -action on the product $\mathcal{B} \times \mathcal{B}$ of two copies of the flag variety. We extend this to an action of G_η by setting

$$(1, \eta) \cdot (B_1, B_2) = (\eta(B_2), \eta(B_1)).$$

We refer to the corresponding quotient stack $(\mathcal{B} \times \mathcal{B})/G_\eta$ as the symmetric flag space.

7.1.3. The global Langlands space. Consider the loop space $\mathcal{L}((\mathcal{B} \times \mathcal{B})/G_\eta)$, and its corresponding $\mathbf{Z}/2\mathbf{Z}$ -rigidification $\mathcal{L}((\mathcal{B} \times \mathcal{B})/G_\eta)_{\text{rigid}}$. Our immediate goal is to spell out what this space is in terms of explicit equations.

First and foremost, by Proposition 4.2, we know that $\mathcal{L}((\mathcal{B} \times \mathcal{B})/G_\eta)_{\text{rigid}}$ is an ordinary stack with trivial derived structure.

Next, the canonical projection $(\mathcal{B} \times \mathcal{B})/G_\eta \rightarrow pt/G_\eta$ induces a projection of rigidified loop spaces

$$\mathcal{L}((\mathcal{B} \times \mathcal{B})/G_\eta)_{\text{rigid}} \rightarrow G_\eta/G.$$

Taking the preimages of the two connected components of G_η/G , we obtain a decomposition of $\mathcal{L}((\mathcal{B} \times \mathcal{B})/G_\eta)_{\text{rigid}}$ into two connected components.

The component of $\mathcal{L}((\mathcal{B} \times \mathcal{B})/G_\eta)_{\text{rigid}}$ above the identity component of G_η/G is canonically isomorphic to the usual global Steinberg space St/G . Both can be identified with the loop space of the usual flag space $(\mathcal{B} \times \mathcal{B})/G$.

We write St^η/G for the second component of $\mathcal{L}((\mathcal{B} \times \mathcal{B})/G_\eta)_{\text{rigid}}$, and refer to it as the global Langlands space. By definition, it consists of paths in the flag space $(\mathcal{B} \times \mathcal{B})/G$ which begin at a pair of flags (B_1, B_2) and end at the pair $(\eta(B_2), \eta(B_1))$.

To make this more explicit, consider the composite map

$$St^\eta/G \rightarrow (\mathcal{B} \times \mathcal{B})/G \rightarrow \mathcal{B}/G$$

given by projection to the first flag. Then the fiber of St^η/G above a fixed flag $B \in \mathcal{B}$ is the ordinary fiber product

$$B \times_G (G_\eta \setminus G)$$

with respect to the inclusion $B \hookrightarrow G$ and the square map $(G_\eta \setminus G) \rightarrow G$. Allowing the flag to vary, we see that $\mathcal{S}t^\eta/G$ is the moduli stack

$$\mathcal{S}t^\eta/G = \{(B \in \mathcal{B}, (g, \delta) \in B \times_G (G_\eta \setminus G))\}/G.$$

Simplifying the notation, we see that $\mathcal{S}t^\eta/G$ is the stack parametrizing pairs

$$\mathcal{S}t^\eta/G = \{B \in \mathcal{B}, \delta \in (G_\eta \setminus G) \text{ such that } \delta^2 \in B\}/G.$$

If we further identify $(G_\eta \setminus G)$ with G itself, then the square map becomes simply

$$g \mapsto \eta(g)g.$$

Thus we may think of $\mathcal{S}t_G^\eta$ as the stack parametrizing pairs

$$\mathcal{S}t^\eta/G = \{B \in \mathcal{B}, g \in G \text{ such that } \eta(g)g \in B\}/G.$$

7.1.4. The Langlands space. Let $H = B/U$ be the universal Cartan, and consider the natural projection $p : \mathcal{S}t^\eta/G \rightarrow H$ given by

$$p(B, \delta) = [\delta^2] \in B/U.$$

We call H the monodromic base, and refer to p as the monodromic projection.

Let \widehat{H}_e be the formal neighborhood of the unit $e \in H$, and define the Langlands space $\mathcal{S}t_e^\eta/G$ to be the inverse image

$$\mathcal{S}t_e^\eta/G = p^{-1}(\widehat{H}_e).$$

7.1.5. The formal Langlands space. The Langlands space $\mathcal{S}t_e^\eta/G$ breaks up into many connected components. To describe this decomposition, consider the space of involutions

$$\mathcal{I} = \{\iota \in (G^\eta \setminus G) \mid \iota^2 = 1\}.$$

For convenience, fix once and for all an element ι in each connected component of \mathcal{I} , and let \mathcal{I}_ι denote the connected component containing ι .

Each ι provides an involution of G by conjugation, and we write $K_\iota \subset G$ for the corresponding fixed point subgroup. By construction, the quotient stack \mathcal{I}_ι/G is isomorphic to the classifying space pt/K_ι .

For each ι , we write \mathcal{N}_ι for the connected component of the image of the projection

$$\mathcal{S}t_e^\eta \rightarrow (G_\eta \setminus G).$$

containing ι . Then we have that $\mathcal{S}t_e^\eta/G$ is the disjoint union of the connected components

$$\mathcal{S}t_e^\iota/G = \{(B \in \mathcal{B}, (g, \delta) \in B \times_G \mathcal{N}_\iota)\}/G.$$

As in the construction of $\mathcal{S}t^\eta/G$, the ordinary fiber product here is with respect to the inclusion $B \hookrightarrow G$ and the square map $\mathcal{N}_\iota \rightarrow G$.

Consider the K_ι -action on the flag variety \mathcal{B} and the canonical embedding

$$e : \mathcal{B}/K_\iota \rightarrow \mathcal{S}t_e^\iota/G \quad e(B) = (B, (1, \iota)).$$

We write $\widehat{\mathcal{S}t}_e^\iota/G$ for the formal neighborhood of the image of e , and refer to it as the formal Langlands space for ι .

Theorem 7.1. *There is a canonical isomorphism*

$$\mathcal{H}(\mathcal{B}/K_\iota) \simeq \widehat{\mathcal{S}t}_e^\iota/G.$$

Proof. First, observe that the left hand side parametrizes pairs

$$\mathcal{H}(\mathcal{B}/K_\iota) = \{(B \in \mathcal{B}, (g, k) \in B \times_G \widehat{K}_\iota)\}/K_\iota$$

where \widehat{K}_ι denotes the formal group of K_ι . Here the ordinary fiber product is with respect to the inclusions of subgroups $B \hookrightarrow G$ and $\widehat{K}_\iota \hookrightarrow G$.

Recall that the Langlands space $\mathcal{S}t_e^\iota/G$ parametrizes pairs

$$\mathcal{S}t_e^\iota/G = \{(B \in \mathcal{B}, (g, \delta) \in B \times_G \mathcal{N}_\iota)\}/G$$

where the ordinary fiber product is with respect to the inclusion $B \hookrightarrow G$ and the square map $\mathcal{N}_\iota \rightarrow G$.

Now we can define a map

$$\mathcal{H}(\mathcal{B}/K_\iota) \rightarrow \widehat{\mathcal{S}t_e^\iota}/G \quad (B, (g, k)) \mapsto (B, (g^2, k\iota)).$$

That this is an isomorphism follows from the constructions and the fact that for any formal group in characteristic zero, the squaring map is an isomorphism. \square

As can be seen explicitly from the proof, Theorem 7.1 does *not* in general provide an S^1 -equivariant isomorphism. Let us compare the universal monodromies of the respective S^1 -actions under the induced quasi-equivalence

$$L_{\text{qcoh}}(\mathcal{H}(\mathcal{B}/K_\iota)) \xrightarrow{\sim} L_{\text{qcoh}}(\widehat{\mathcal{S}t_e^\iota}/G).$$

By construction, we have an identity

$$m_{\mathcal{S}t}^\iota = m_{\mathcal{H}}^\iota \circ \varphi_\iota$$

where $m_{\mathcal{S}t}^\iota$ is the universal monodromy of the right hand side, $m_{\mathcal{H}}^\iota$ is that of the left hand side, and φ_ι is the automorphism of the identity functor given by the element $\iota \in G_\eta \setminus G$.

Although we will not use it, it is also worth pointing out that since $\iota^2 = 1$, the squared S^1 -actions on each side in fact coincide.

7.2. Harish-Chandra modules. Let $\mathfrak{U}(\mathfrak{g})$ denote the universal enveloping algebra of the Lie algebra \mathfrak{g} . For a fixed involution ι of G with fixed-point subgroup K_ι , consider the Harish-Chandra pair $(\mathfrak{U}(\mathfrak{g}), K_\iota)$, and the corresponding category of Harish-Chandra modules with trivial infinitesimal character. By definition, these are modules over the Harish-Chandra pair $(\mathfrak{U}(\mathfrak{g}), K_\iota)$ which are finitely generated over $\mathfrak{U}(\mathfrak{g})$ and on which the center $\mathfrak{Z}(\mathfrak{g}) \subset \mathfrak{U}(\mathfrak{g})$ acts via the trivial character.

By Beilinson-Bernstein localization, the abelian category of Harish Chandra modules for $(\mathfrak{U}(\mathfrak{g}), K_\iota)$ with trivial infinitesimal character is equivalent to the abelian category of K_ι -equivariant \mathcal{D} -modules on \mathcal{B} . One should beware that the naive corresponding dg derived category of Harish-Chandra modules is *not* equivalent to the natural K_ι -equivariant dg derived category of \mathcal{D} -modules on \mathcal{B} . It is the latter category that plays a more prominent role in representation theory (for example, by the work of [ABV, So] as parameters for certain representations), and thus we will proceed with it in mind as our model for the dg derived category of Harish-Chandra modules. Alternatively, one could take the appearance of Harish-Chandra modules as purely motivational, and understand that it is the K_ι -equivariant dg derived category of \mathcal{D} -modules on \mathcal{B} that we are most interested in.

Now we arrive at our goal of identifying localized S^1 -equivariant coherent sheaves on the component of the Langlands space $\mathcal{S}t_e^\iota/G$.

Theorem 7.2. *There is a canonical quasi-equivalence of dg derived categories*

$$L_{\text{coh}}(\mathcal{S}t_e^\iota/G)^{S^1}_{\text{per}} \xrightarrow{\sim} L_{\text{coh}}(\mathcal{D}_{\mathcal{B}/K_\iota}) \otimes_{\mathbf{C}} \mathbf{C}[u, u^{-1}].$$

The proof is completely analogous to arguments appearing in Section 6. First, as in Lemma 6.4, one checks that the natural restriction map

$$L_{\text{coh}}(\mathcal{S}t_e^\iota/G) \rightarrow L_{\text{coh}}(\widehat{\mathcal{S}t_e^\iota}/G)$$

is an S^1 -equivariant equivalence. Then as in Theorem 6.9, one uses Theorem 7.1 and the discussion thereafter to deduce an equivalence of localized S^1 -equivariant categories. We leave it to the interested reader to trace through the arguments.

7.3. General monodromicities. In this section, we establish results for a general monodromic parameter analogous to those of the previous sections.

Recall that we have the monodromic projection $p : \mathcal{S}t^\eta/G \rightarrow H$, and that the Langlands space $\mathcal{S}t^\eta/G$ is defined to be the inverse image under p of the formal group \widehat{H}_e .

More generally, given $\alpha \in H$, we denote its formal neighborhood by \widehat{H}_α , and define the monodromic Langlands space $\mathcal{S}t_\alpha^\eta/G$ to be the inverse image

$$\mathcal{S}t_\alpha^\eta/G = p^{-1}(\widehat{H}_\alpha).$$

Our aim here is to give a loop space interpretation of coherent sheaves on $\mathcal{S}t_\alpha^\eta/G$.

As with the case already considered when α is the identity, we have the following picture. The monodromic Langlands space $\mathcal{S}t_\alpha^\eta/G$ breaks up into many connected components. To describe this decomposition, fix a semisimple representative $\tilde{\alpha} \in G$, and consider the space of elements

$$\mathcal{I}_\alpha = \{\iota \in (G^\eta \setminus G) \mid \iota^2 = \tilde{\alpha}\}.$$

For convenience, fix once and for all an element ι in each connected component of \mathcal{I}_α , and let $\mathcal{I}_{\alpha,\iota}$ denote the connected component containing ι .

Let $G(\alpha) \subset G$ be the centralizer of $\tilde{\alpha}$. Each ι provides an involution of $G(\alpha)$ by conjugation, and we write $K(\alpha)_\iota \subset G(\alpha)$ for the corresponding fixed-point subgroup. By construction, the quotient stack $\mathcal{I}_{\alpha,\iota}/G(\alpha)$ is isomorphic to the classifying space $pt/K(\alpha)_\iota$.

For each ι , we write $\mathcal{N}_{\alpha,\iota}$ for the connected component of the image of the projection

$$\mathcal{S}t_\alpha^\eta \rightarrow (G_\eta \setminus G).$$

containing ι . Then we have that $\mathcal{S}t_\alpha^\eta/G$ is the disjoint union of the connected components

$$\mathcal{S}t_\alpha^\iota/G = \{(B \in \mathcal{B}, (g, \delta) \in B \times_G \mathcal{N}_{\alpha,\iota})\}/G.$$

As in the construction of $\mathcal{S}t^\eta/G$, the ordinary fiber product here is with respect to the inclusion $B \hookrightarrow G$ and the square map $\mathcal{N}_{\alpha,\iota} \rightarrow G$.

Let $\mathcal{B}(\alpha)$ be the flag variety of $G(\alpha)$, and consider the $K(\alpha)_\iota$ -action on $\mathcal{B}(\alpha)$. Given a Borel $B(\alpha) \in \mathcal{B}(\alpha)$, there is a unique Borel $B \in \mathcal{B}$ containing $B(\alpha)$ such that the class of $\tilde{\alpha}$ in the universal Cartan $H = B/U$ is equal to α . Thus we have a canonical embedding

$$e : \mathcal{B}(\alpha)/K(\alpha)_\iota \rightarrow \mathcal{S}t_\alpha^\iota/G \quad e(B(\alpha)) = (B, (\tilde{\alpha}, \iota))$$

We write $\widehat{\mathcal{S}t}_\alpha^\iota/G$ for the formal neighborhood of the image of e , and refer to it as the formal monodromic Langlands space for ι .

The following is easily checked as in Theorem 7.1 when α is trivial. We leave further details to the interested reader.

Theorem 7.3. *There is a canonical isomorphism*

$$\mathcal{H}(\mathcal{B}(\alpha)/K(\alpha)_\iota) \simeq \widehat{\mathcal{S}t}_\alpha^\iota/G.$$

As in Theorem 7.1, the identification of Theorem 7.3 does *not* in general provide an S^1 -equivariant isomorphism. Let us compare the universal monodromies of the respective S^1 -actions under the induced quasi-equivalence

$$L_{\text{qcoh}}(\mathcal{H}(\mathcal{B}(\alpha)/K(\alpha)_\iota)) \xrightarrow{\sim} L_{\text{qcoh}}(\widehat{\mathcal{S}t}_\alpha^\iota/G).$$

Under this identification, we have an identity

$$m_{\mathcal{S}t_\alpha}^\iota = m_{\mathcal{H}_\alpha}^\iota \circ \varphi_\iota$$

where $m_{\mathcal{S}t_\alpha}^\iota$ is the universal monodromy of the right hand side, $m_{\mathcal{H}_\alpha}^\iota$ is that of the left hand side, and φ_ι is the automorphism of the identity functor given by the element $\iota \in G_\eta \setminus G$.

We arrive at our goal of indentifying localized S^1 -equivariant coherent sheaves on the component of the monodromic Langlands space $\mathcal{S}t_\alpha^\iota/G$.

Theorem 7.4. *There is a canonical quasi-equivalence of dg derived categories*

$$L_{\text{coh}}(\mathcal{S}t_\alpha^\iota/G)_{\text{per}}^{S^1} \xrightarrow{\sim} L_{\text{coh}}(\mathcal{D}_{\mathcal{B}_\alpha/K_{\alpha,\iota}}) \otimes_{\mathbf{C}} \mathbf{C}[u, u^{-1}].$$

The proof is completely analogous to that of Theorem 7.2 and follows arguments appearing in Section 6. First, as in Lemma 6.4, one checks that the natural restriction map

$$L_{\text{coh}}(\mathcal{S}t_\alpha^\iota/G) \rightarrow L_{\text{coh}}(\widehat{\mathcal{S}t}_\alpha^\iota/G)$$

is an S^1 -equivariant equivalence. Then as in Theorem 6.9, one uses Theorem 7.3 and the discussion thereafter to deduce an equivalence of localized S^1 -equivariant categories. We leave it to the interested reader to trace through the arguments.

8. APPENDIX: DERIVED STACKS

In what follows, all schemes, stacks, etc are assumed to be over a field k of characteristic 0. Unless otherwise stated, all rings are assumed to be commutative with unit.

8.1. Some motivation. Our main objects of study are *derived stacks* in the sense of [To2, L1]. Before recalling what is meant by a derived stack, we informally review some motivation for their introduction.

As we will outline momentarily, derived stacks are a broad context for dealing with natural questions in algebraic geometry. But perhaps the best motivation for considering derived stacks is not what they bring to algebraic geometry, but that they bring algebraic geometry to other areas. One of the most exciting examples of this is the unity of stable homotopy theory and derived formal groups in the language of the “brave new algebraic geometry” of E_∞ -ring spectra. In this paper, we discuss how derived algebraic geometry provides a natural language for discussing basic objects of geometric representation theory. In particular, Steinberg varieties and \mathcal{D} -modules may be described as derived stacks and quasicoherent sheaves on derived stacks respectively.

Our starting point is the study of schemes, or more generally Artin algebraic spaces. (The latter are obtained from the former by allowing étale equivalence relations.) Throughout the discussion, we will think of such geometric objects via their functors of points. Thus by an algebraic space, we will mean a functor

$$\mathcal{F} : \text{Rings} \rightarrow \text{Sets}$$

that is a sheaf in the étale topology, and admits an étale atlas (a representable étale surjection $U \rightarrow \mathcal{F}$ where U is a scheme; the étale equivalence relation on U is given by the fiber product $U \times_{\mathcal{F}} U$).

Even if one is only interested in schemes or algebraic spaces, natural geometric constructions produce objects which lie beyond their definition. Two fundamental examples worth keeping in mind are the problems of finding universal families and forming intersections. The difficulties of the first are well-known: it is impossible to find a space that parametrizes familiar objects such as curves or vector bundles. In other words, there is an insurmountable obstruction to constructing a space whose points are in natural bijection with the objects under consideration. The source of the difficulty is that such objects come with large amounts of symmetry; no space will be rich enough to parametrize all possible twisted families. To overcome this, there is the theory of algebraic stacks: one expands the notion of a space to include functors

$$\mathcal{F} : \text{Rings} \rightarrow \text{Groupoids}.$$

Natural notions of representability include Deligne-Mumford stacks (where \mathcal{F} is a sheaf in the étale topology and admits an étale atlas) and Artin stacks (where \mathcal{F} is a sheaf in the faithfully flat quasi-compact topology and admits a smooth atlas). Thus rather than trying to wrestle with the automorphisms of objects, we accept their presence and parametrize the objects and their symmetries simultaneously. It is worth commenting that if one continues with such considerations, one finds the same deficiency in restricting to functors with values in groupoid. A hint of higher stacks appears: one should rather consider functors with values in arbitrary simplicial sets.

A similarly well-known difficulty is inherent in the intersection theory of schemes. Basic facts about intersections of schemes fail as soon as the intersection is degenerate. The usual solution is to recognize that the discrepancy may be accounted for by higher homological invariants of the intersection. Rather than only considering the naive tensor product which defines the scheme-theoretic intersection, we should also keep track of the higher Tor terms as well. Considering such derived functors and their underlying differential graded avatars has become central in the study of coherent sheaves. But one aspect of the situation has only come into focus relatively recently: when derived constructions (such as the tensor product of rings) produce a differential graded ring, we should continue to regard this as kind of generalized scheme. The resulting theory of derived schemes considers functors from differential graded rings to simplicial sets; more generally (though equivalent to the differential graded theory in characteristic zero), one studies functors of the form

$$\mathcal{F} : \text{Simplicial rings} \rightarrow \text{Simplicial sets}.$$

Here the appearance of simplicial sets follows naturally from the fact that we have introduced simplicial rings. For example, since morphisms between simplicial rings are enriched over simplicial sets, representable functors naturally take values in simplicial sets. (But it is worth mentioning that the derived schemes that are the building blocks of the theory to be discussed will continue to take ordinary rings to ordinary sets.) Although for the time being we are postponing any formal details, it is important to comment that what we care about here is not the simplicial structure on our rings and sets, but rather only homotopically meaningful properties. Thus via the geometric realization of simplicial sets, we may equivalently consider functors on topological rings with values in topological spaces. It is worth mentioning that one may also take the perspective that a derived scheme is a kind of “locally ringed space” with structure sheaf a topological ring.

Finally, we arrive at the theory of derived stacks by passing to the natural level of generality implicit in the above theories of stacks and derived schemes. Namely, we continue to consider functors of the form

$$\mathcal{F} : \text{Simplicial rings} \rightarrow \text{Simplicial sets}$$

but no longer make such strict assumptions about the vanishing of higher homotopy groups of the functors when evaluated on ordinary rings. As before, we only care about the homotopic properties of test objects and functor values, and so could equivalently consider functors from topological rings to topological spaces. In the next section, we will recall the setting of quasicategories which keeps track of the correct amount of structure. It is worth mentioning one more motivation for considering derived stack. If we have accepted that stacks are fundamental objects, then we would hope our working framework would encompass natural constructions involving them. As a basic example, one can see that the cotangent bundle of something as simple as a classifying stack is already a derived stack.

8.2. Basic terminology. Our aim in this section is to review some of the basic terminology in the theory of derived stacks. It is a formidable task to come to terms with the intricate foundation of this theory. Fortunately, there are several excellent sources to which we may refer the reader [To2, L1]. In the following, we content ourselves with introducing the main objects and their most relevant attributes.

One challenge of introducing derived stacks is their unfamiliar categorical underpinnings. A natural setting for their discussion is not category theory strictly speaking but higher category theory in the form of topological categories. In other words, we should work in the context of categories enriched over topological spaces: the morphisms between objects are topological spaces and the composition maps are continuous. While this is a correct approach (and equivalent to the approach adopted below), it is often unnecessarily restrictive in practice. To provide more elbow room, we should rather work in some more flexible version of $(\infty, 1)$ -categories such as quasicategories (there is also the alternative framework of Segal categories [ToVe1], see [Ber] for a comparison between the different frameworks). To give the definition of quasicategories is easy: they are certain simplicial sets sometimes called weak Kan complexes. Let Δ^n denote the standard n -simplex, and $\Lambda_i^n \hookrightarrow \Delta^n$ the “inner horn” obtained by deleting the interior open n -simplex and the i th $(n - 1)$ -dimensional face.

Definition 8.1. A quasicategory is a simplicial set K satisfying the following extension property: for any $0 < i < n$, any map $\Lambda_i^n \rightarrow K$ extends to a map $\Delta^n \rightarrow K$.

The theory of quasicategories is well documented in the literature, and there are many good sources for the interested reader (see in particular [Jo] and the survey [Ber], in addition to the book [L2]). Rather than recalling any further formal properties, it may be more meaningful to try to spell out what motivates the definition. Roughly speaking, one wants the notion of a category whose morphisms are topological spaces and whose compositions and associativity properties are defined up to coherent homotopies. For a quasicategory, the objects are given by its 0-simplices, and the n -morphisms by its n -simplices. Though there is no explicit mention of an associative composition in the above definition, the extension property is exactly what is needed to define such structure up to coherent homotopies. Finally, given a topological category, one may construct a quasicategory by taking its topological nerve (which is by definition the simplicial nerve of its singular complex). This sets up an equivalence between the theory of topological categories and that of quasicategories.

A common way that quasicategories arise is via the localization of simplicial model categories. One may think of many quasicategories as fitting into a sequence

$$\text{Simplicial model category } \mathcal{C} \rightarrow \text{Quasicategory } N(\mathcal{C}^\circ) \rightarrow \text{Homotopy category } h\mathcal{C}$$

where the quasicategory $N(\mathcal{C}^\circ)$ is the simplicial nerve of the subcategory $\mathcal{C}^\circ \subset \mathcal{C}$ of fibrant-cofibrant objects. More generally, one can associate to any model category an underlying quasicategory by inverting the weak equivalences in the appropriate sense. For example, starting

from the categories of (compactly generated Hausdorff) topological spaces and simplicial sets with their usual model structures, we obtain canonically equivalent quasicategories which we will identify, denote by \mathcal{S} , and refer to as the quasicategory of spaces. Note that passing to the homotopy category is often too drastic: there is not enough structure in order to make usual constructions. To make an analogy, we sometimes think of the following toy model of the above sequence

$$\text{Based vector spaces} \rightarrow \text{Vector spaces} \rightarrow \text{Dimensions of vector spaces.}$$

While it is often best to consider vector spaces with no preferred basis, standard constructions can not be made at the level of their dimensions.

With the preceding discussion in hand, we now proceed to recall the definition of derived stacks. As informally discussed in the previous section, we will think of derived stacks in terms of their functors of points. Thus the first order of business is to describe what our test objects are and where our functors take their values.

Let \mathcal{CA}_k denote the category of commutative unital k -algebras, and let \mathcal{CA}_k^Δ denote its simplicial category. We endow the latter with the structure of a simplicial model category in which the weak equivalences and fibrations are weak equivalences and fibrations on the underlying simplicial sets. Let \mathcal{SCA}_k denote the quasicategory obtained from \mathcal{CA}_k by considering its fibrant-cofibrant objects. One refers to objects of its opposite quasicategory as affine derived schemes. It is possible to speak of étale and smooth morphisms between affine derived schemes, and hence in particular the étale topology on \mathcal{SCA}_k . Recall that \mathcal{S} denotes the quasicategory of spaces obtained from the simplicial model category of simplicial sets, or equivalently (compactly generated Hausdorff) spaces, by considering fibrant-cofibrant objects.

Definition 8.2. A derived scheme over k is a functor

$$\mathcal{F} : \mathcal{SCA}_k \rightarrow \mathcal{S}$$

that is a sheaf in the étale topology, and admits an étale atlas $U \rightarrow \mathcal{F}$ where U is an affine derived scheme.

The reader will notice that the definition is more akin to that of a Deligne-Mumford stack than an ordinary scheme. This turns out to be a more natural notion from the perspective of locally ringed spaces, or more accurately ringed topos theory. Namely, one may alternatively define a derived scheme to be an ∞ -topos \mathcal{X} equipped with a \mathcal{SCA}_k -valued structure sheaf \mathcal{O} satisfying the following representability: there is a collection of objects $U_\alpha \in \mathcal{X}$ such that the map $\coprod_\alpha U_\alpha \rightarrow 1_{\mathcal{X}}$ is surjective, and each ∞ -topos $\mathcal{X}_{/U_\alpha}$ with structure sheaf $\mathcal{O}_{\mathcal{X}}|_{U_\alpha}$ is equivalent to the spectrum of some simplicial commutative ring. The allowable gluings of the categorical setting of topoi naturally lead to the generality of Deligne-Mumford stacks. To arrive at the derived notion of algebraic space, we simply impose the following condition on the gluings.

Definition 8.3. A derived algebraic space over k is a derived scheme \mathcal{F} such that $\mathcal{F}(A)$ is discrete whenever A is discrete.

Now to pass to arbitrary derived Artin stacks, we relax the representability assumption of a derived algebraic space. (Note that we are already considering functors with values in the quasicategory of spaces \mathcal{S} so there is no need to generalize anything in this direction.) We will use the term derived stack to refer to any functor

$$\mathcal{F} : \mathcal{SCA}_k \rightarrow \mathcal{S}$$

that is a sheaf in the étale topology. Let us consider what kind of representability we could allow for an atlas $p : U \rightarrow \mathcal{F}$. Since all of our previous objects (specifically derived algebraic spaces) admit atlases of affine derived schemes, we will keep to this here as well and assume U is an affine derived scheme. But now it makes sense to allow p to be a smooth relative derived algebraic space rather than only an étale map. One calls sheaves \mathcal{F} that admits such a presentation 1-Artin stacks. Now of course, we could iterate this definition and allow étale sheaves \mathcal{F} that admit atlases $p : U \rightarrow \mathcal{F}$ where U is an affine derived scheme, but p is a smooth relative 1-Artin stack. One calls sheaves \mathcal{F} that admits such a presentation 2-Artin stacks. And so on: the inductive story is encapsulated in the following definition.

Definition 8.4. Let \mathcal{D} be the quasicategory of derived stacks whose objects are \mathcal{S} -valued étale sheaves on \mathcal{SCA}_k . In what follows, let $X \rightarrow Y$ be a map of such sheaves, and let T be an arbitrary affine derived test scheme.

A morphism $p : X \rightarrow Y$ is a relative 0-Artin stack if for any map $T \rightarrow Y$, the base change $T \times_Y X$ is a derived algebraic space. Such a map p is said to be smooth if the induced map $T \times_Y X \rightarrow T$ is smooth as a map of derived schemes.

For $n > 0$, a morphism $p : X \rightarrow Y$ is a relative n -Artin stack if for any map $T \rightarrow Y$, there exists an affine derived scheme U , and a smooth surjection $U \rightarrow T \times_Y X$ which is a relative $(n - 1)$ -Artin stack. Such a map p is said to be smooth if the induced map $U \rightarrow T$ is smooth.

A derived n -Artin stack X is a relative derived n -Artin stack $X \rightarrow \mathrm{Spec}(k)$. A derived Artin stack is a derived n -Artin stack for some $n \geq 0$.

It is worth remarking that the basic building blocks of derived Artin stacks (according to the above definition) are *affine* derived schemes as opposed to all derived schemes. In particular, not all derived schemes are derived stacks in the above sense. Rather, one can check that for $n \geq 1$, a derived scheme \mathcal{F} is a derived n -stack if and only if for all discrete test rings A , the homotopy groups of $\mathcal{F}(A)$ vanish in degrees greater than or equal to n .

One final issue to comment upon before wrapping up this survey is our choice of simplicial commutative rings as coefficients. There are several possible generalizations of the ordinary category of discrete rings which one might consider when defining affine derived schemes and hence derived stacks. In the preceding discussion, we have worked with the quasicategory \mathcal{SCA}_k of simplicial commutative k -algebras, or equivalently topological k -algebras. To an algebraically minded person, this world may feel very far from the intuitions of discrete rings. Thus it is worth pointing out that in characteristic zero, in place of topological k -algebras one may instead consider the category of commutative differential graded k -algebras. It admits a model structure in which the weak equivalences are simply the quasiisomorphisms, and the cofibrations are retracts of iterated cell attachments. We write \mathcal{DGA}_k for the underlying quasicategory. The Dold-Kan theorem provides a canonical fully faithful embedding

$$\mathcal{SCA}_k \hookrightarrow \mathcal{DGA}_k.$$

Its essential image consists of connective objects (those objects whose cohomology vanishes in negative degrees). Thus in characteristic zero, we may think of the quasicategory of affine derived schemes as opposite to that of connective differential graded k -algebras. In the main text of this paper, we freely interpolate between the simplicial and differential graded points of view.

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